Princeton University
Department of Operations Research
and Financial Engineering

ORF 307
Optimization

Practice Second Midterm

Closed book. No computers. Calculators allowed (but not needed).

You are permitted to use a one-page two-sided cheat sheet.

Please return the exam questions and your cheat sheet with your exam booklet.

---

(1) Consider the following linear programming problem:

$$\begin{array}{lrcrcrcrcrcl} \text{maximize} & x_1 & + & 2x_2 & + & 4x_3 & + & 8x_4 & + & 16x_5 & & \\ \text{subject to} & x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & \leq & 2 \\ & & & 5x_2 & - & 3x_3 & - & 2x_4 & - & x_5 & \leq & 3 \\ & & & & & & x_1, & x_2, & x_3, & x_4, \; x_5 & \geq & 0. \end{array}$$

Consider the situation in which x_1 and x_5 are basic and all other variables are non-basic. Write down:

(a) B,
(b) N,
(c) b,
(d) $c_{\mathcal{B}}$,
(e) $c_{\mathcal{N}}$,
(f) $B^{-1}N$,
(g) $x^*_{\mathcal{B}} = B^{-1}b$,
(h) $\zeta^* = c_{\mathcal{B}}^T B^{-1}b$,
(i) $z^*_{\mathcal{N}} = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}$,
(j) the dictionary corresponding to this basis.

$B = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$ $N = \begin{bmatrix} 2 & 3 & 4 & 1 & 0 \\ 5 & -3 & -2 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$c_N = [2 \; 4 \; 8 \; 0 \; 0]^T$ $c_B = [1 \; 16]^T$

$B^{-1} = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$ $B^{-1}N = \begin{bmatrix} 27 & -12 & -6 & 1 & 5 \\ -5 & 3 & 2 & 0 & -1 \end{bmatrix}$

$x^*_B = \begin{bmatrix} 17 \\ -3 \end{bmatrix}$ $\zeta^* = -31$

$z^*_N = \begin{bmatrix} -53 \\ 36 \\ 26 \\ 1 \\ -11 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 8 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -55 \\ 32 \\ 18 \\ 1 \\ -11 \end{bmatrix}$

$$\boxed{\begin{array}{rcl} \zeta &=& -31 + 55x_2 - 32x_3 - 18x_4 - w_1 + 11w_2 \\ x_1 &=& 17 - 27x_2 + 12x_3 + 6x_4 - w_1 - 5w_2 \\ x_5 &=& -3 + 5x_2 - 3x_3 - 2x_4 + w_2 \end{array}}$$

(2) Use the parametric self-dual simplex method to solve the following problem:

$$\begin{array}{lrcrcr} \text{maximize} & 3x_1 & - & x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & 1 \\ & -x_1 & + & x_2 & \leq & -4 \\ & & & x_1,\ x_2 & \geq & 0. \end{array}$$

$$\begin{array}{llll} \zeta = & & (3-\mu)x_1 & +(-1-\mu)x_2 \\ \hline w_1 = & 1+\mu & -x_1 & +x_2 \\ w_2 = & -4+\mu & +x_1 & -x_2 \end{array}$$

$\Rightarrow 4 \le \mu < \infty \Rightarrow$ w_2 leaves, x_1 enters

$$\begin{array}{llll} \zeta = & 12+\cdots & +(3-\mu)w_2 & +(2-2\mu)x_2 \\ \hline w_1 = & -3+2\mu & -w_2 & \\ x_1 = & 4-\mu & +w_2 & +x_2 \end{array}$$

$\Rightarrow 3 \le \mu \le 4 \Rightarrow$ w_2 enters, w_1 leaves

$$\begin{array}{llll} \zeta = & 3+\cdots & +(-3+\mu)w_1 & +(2-2\mu)x_2 \\ \hline w_2 = & -3+2\mu & -w_1 & \\ x_1 = & 1+\mu & -w_1 & +x_2 \end{array}$$

$\Rightarrow 3/2 \le \mu \le 3 \Rightarrow$ w_2 leaves, no one enters. Primal Infeas!

(3) Two players simultaneously throw out two or three fingers and call out their guess as to what the total sum of the outstretched fingers will be. If a player guesses right, but her opponent does not, then she receives from her opponent payment equal to her opponent's guess. In all other cases, it is a draw.

$$A = \begin{array}{c} \\ 2,4 \\ 2,5 \\ 3,5 \\ 3,6 \end{array} \begin{array}{c} \begin{array}{cccc} 2,4 & 2,5 & 3,5 & 3,6 \end{array} \\ \left[\begin{array}{cccc} 0 & -5 & 4 & 0 \\ 5 & 0 & 0 & -6 \\ -4 & 0 & 0 & 5 \\ 0 & 6 & -5 & 0 \end{array}\right] \end{array}$$

(a) List the pure strategies for this game. → A pure strategy consists of two integers: number of fingers thrown and guess for sum. Possibilities are: (2,4), (2,5), (3,5), (3,6)
(b) Write down the payoff matrix for this game.
(c) Formulate the row player's problem as a linear programming problem. (*Hint: Recall that the row player's problem is to minimize the maximum expected payout.*) ← As in book
(d) What is the value of this game? 0. The game is symmetric. Neither has an edge.

$$f(x) = \sum_{i=1}^{m} \left(|x-b_i| + \tfrac{1}{2}(x-b_i)\right)$$

$$f'(x) = \#\{i : x > b_i\} - \underbrace{\#\{i : x < b_i\}}_{m - \#\{i : x > b_i\}} + \frac{m}{2} = 0$$

$$2\#\{i : x > b_i\} = \frac{m}{2}$$

$$\#\{i : x > b_i\} = \frac{m}{4}$$

1^{st} quartile!

(4) (a) Given m real numbers $b_1, b_2, \ldots, b_m$ as input, use calculus to solve

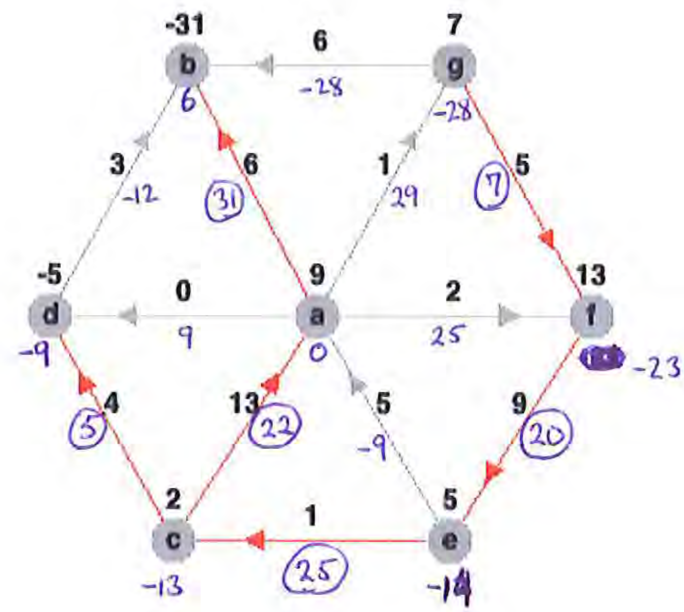
$$\text{minimize} \sum_{i=1}^{m} \left(|x - b_i| + \frac{1}{2}(x - b_i) \right).$$

(b) Formulate the problem as a linear programming problem.

$$\min \sum_{i=1}^{m} t_i + \frac{m}{2}x \left(-\sum_{i=1}^{m} b_i\right) \leftarrow \text{constants!}$$

$$x - b_i \le t_i$$
$$-t_i \le x - b_i$$
$$t_i \ge 0$$
$$x \text{ free}$$

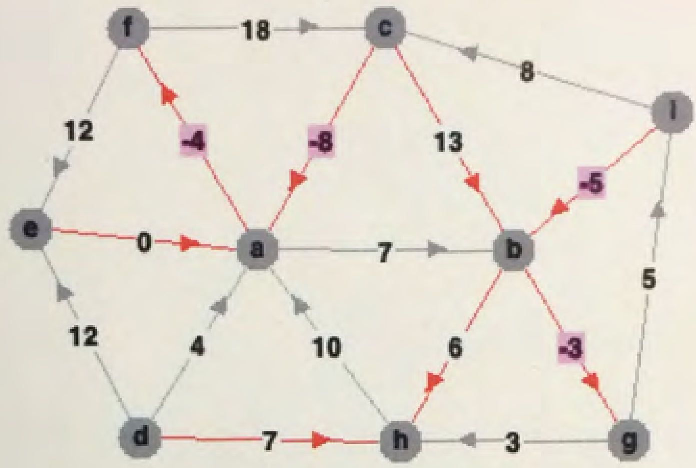
(5) *(11 pts)* Consider the following network flow problem:



The numbers above the nodes are supplies (negative values represent demands) and numbers shown above the arcs are unit shipping costs. The darkened arcs form a spanning tree.

(a) Compute primal flows for each tree arc.
(b) Compute dual variables for each node.
(c) Compute dual slacks for each nontree arc.

(6) Consider the tree solution for the following minimum cost network flow problem:



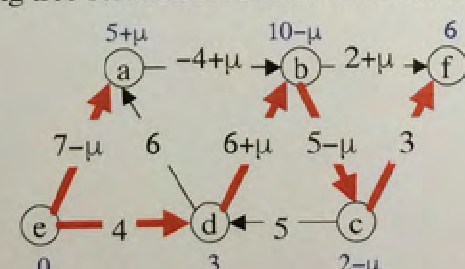
The numbers on the tree arcs represent primal flows while numbers on the nontree arcs are dual slacks.

(a) Using the largest–coefficient rule in the dual network simplex method, what is the leaving arc? Leaving arc: (c, a)

(b) What is the entering arc? Entering arc: (a, b)

(c) After *one* pivot, what is the new tree solution? See p. 6

(7) Consider the following tree solution for a minimum cost network flow problem:



$-4+\mu \geq 0 \Rightarrow \mu \geq 4$
$2+\mu \geq 0 \Rightarrow \mu \geq -2$
$7-\mu \geq 0 \Rightarrow \mu \leq 7$
$6+\mu \geq 0 \Rightarrow \mu \geq -6$
$5-\mu \geq 0 \Rightarrow \mu \leq 5$

$\Downarrow$

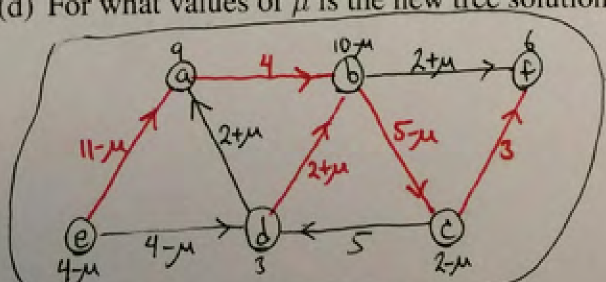
$4 \leq \mu \leq 5$

(a) For what values of μ is this tree solution optimal?

(b) What are the entering and leaving arcs? Entering: (a, b), Leaving (e, d)

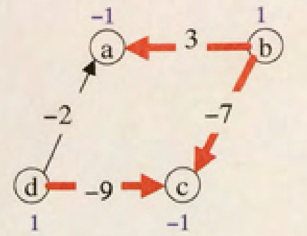
(c) After *one* pivot, what is the new tree solution?

(d) For what values of μ is the new tree solution optimal?



$2+\mu \geq 0 \Rightarrow \mu \geq -2$
$11-\mu \geq 0 \Rightarrow \mu \leq 11$
$2+\mu \geq 0 \Rightarrow \mu \geq -2$
$2+\mu \geq 0 \Rightarrow \mu \geq -2$
$5-\mu \geq 0 \Rightarrow \mu \leq 5$
$4-\mu \geq 0 \Rightarrow \mu \leq 4$
$\Rightarrow -2 \leq \mu \leq 4$

(8) Consider the following minimum cost network flow problem



As usual, the numbers on the arcs represent the flow costs and numbers at the nodes represent supplies (demands are shown as negative supplies). The arcs shown in bold represent a spanning tree. If the solution corresponding to this spanning tree is optimal prove it, otherwise find an optimal solution using this tree as the initial spanning tree.

See next page

(9) Write an AMPL model to solve the following problem:

$$\min_{0 \le x \le \pi} \left(\frac{x^2}{2} - \sin(x) \right).$$

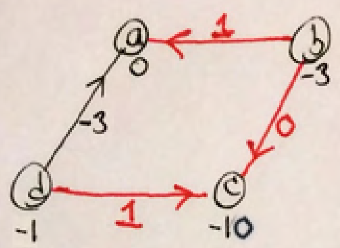
See next page

(10) Graph the following integer programming problem:

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & 5x_2 & & \\ \text{subject to} & -4x_1 & + & 3x_2 & \le & 6 \\ & 3x_1 & + & 2x_2 & \le & 18 \\ & & & x_1,\ x_2 & \ge & 0 \quad \text{and integer.} \end{array}$$

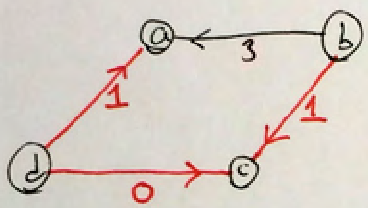
Apply the branch-and-bound procedure, graphically solving each linear programming problem encountered. Interpret the branch-and-bound procedure graphically as well.

⑧



Entering Arc: (d,a)

Leaving Arc: (a, b)

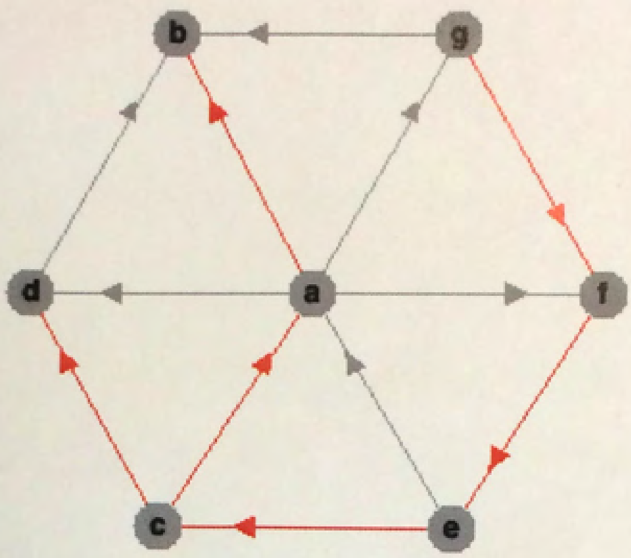


Optimal !

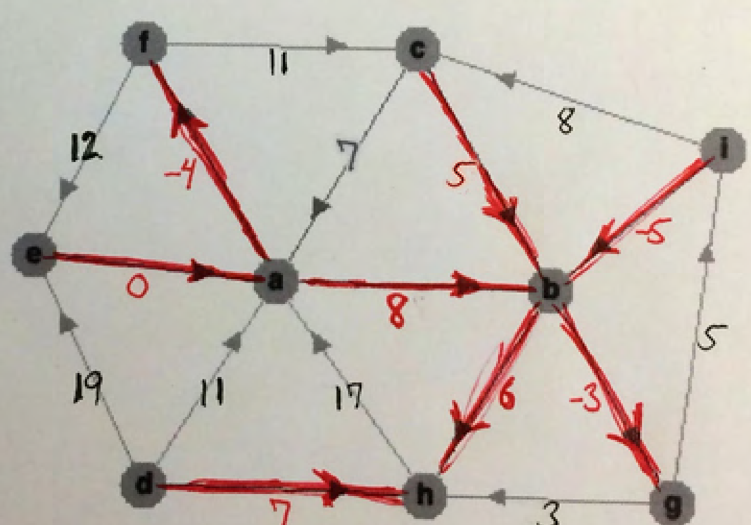
⑨

```
param pi := 3.14159;
var x ≥ 0, ≤ pi;
minimize myobjective: (x^2)/2 - sin(x);
solve;
display x;
```

You may use these "blank" networks to "draw" your solutions to problems 5 and 6.



Network in Problem 5



Network in Problem 6