ORF 245 Fundamentals of Statistics
Chapter 5
Limit Theorems

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Fall 2014

Slides last edited on December 2, 2014

Law of Large Numbers

Let $X_1, X_2, \ldots, X_i, \ldots$ be a sequence of independent identically distributed random variables with mean μ and variance σ^2. Let

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Compute the mean and variance of $\bar{X}_n$:

$$\mathbb{E}(\bar{X}_n) = \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(X_i) = \frac{1}{n} \sum_{i=1}^{n} \mu = \mu$$

$$\mathsf{Var}(\bar{X}_n) = \mathsf{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} \mathsf{Var}(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 = \sigma^2/n$$

The *Law of Large Numbers* says that

$$\lim_{n \to \infty} \bar{X}_n = \mu$$

Technically speaking there are two versions of this theorem: the strong version and the weak version. Such a distinction is important in more advanced classes. We'll ignore it.

Monte Carlo Integration

Suppose we want to integrate some function over some interval. For example, suppose that the function is $f(x) = 1/x$ and the interval is $[1, 4]$:

$$I = \int_1^4 \frac{1}{x} dx$$

There are various methods. First choice would be to do an explicit calculation:

$$I = \int_1^4 \frac{1}{x} dx = \ln(4) - \ln(1) = \ln(4)$$

But, is this really "explicit"? What is the numerical value of $\ln(4)$? Saying there is a button for it on your calculator is not an acceptable answer. Someone at the company that made your calculator had to code up an *algorithm* to do the computation.

In your calculus class, you learned how to approximate integrals using various methods of discretization: the Midpoint Rule, the Trapezoidal Rule, Simpsons Rule, etc. Your calculator uses one of these methods.

Monte Carlo Integration – Continued

Here's another method: generate independent uniformly distributed random variables, $X_1, X_2, \ldots, X_n$ on $[1, 4]$ and compute

$$I_n = \frac{1}{n} \sum_{i=1}^{n} \frac{3}{X_i}$$

The answer is a random variable. But, as n tends to infinity, this random variable converges to the mean value of $3/X$:

$$I_n \longrightarrow \mathbb{E}\left(\frac{3}{X}\right) = \int_1^4 \frac{3}{x}\frac{1}{3}dx = \int_1^4 \frac{1}{x}dx$$

```
% Monte Carlo
n = 300;
X = random('unif', 1, 4, [1 n]);
I = (4-1)*sum(1./X)/n;
sprintf('%10.7f', I)

1.4554425
```

```
% Numerical Integration
x = ((101:400)-0.5)/100;
dx = 1/100;
I = sum(1./x)*dx;
sprintf('%10.7f', I)

1.3862905
```

```
% Matlab's Answer


I = log(4);
sprintf('%10.7f', I)

1.3862944
```

With $n = 3{,}000{,}000$, Monte Carlo gives 1.3862486 (five sig. figs.).

Conclusion: Monte Carlo is a *method of last resort*.

Central Limit Theorem

Let $X_1, X_2, \ldots$ be a sequence of *independent identically distributed (iid)* random variables having mean μ and variance σ^2. Let

$$S_n = \sum_{i=1}^{n} X_i$$

We know that the expected value of a sum is the sum of the expected values and that the variance of a sum of *independent* r.v.s' is the sum of the variances. Hence,

$$\mathbb{E}(S_n) = n\mu \qquad \text{and} \qquad \text{Var}(S_n) = n\sigma^2$$

Therefore, the random variable

$$T_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

has mean zero and variance one.
The *Central Limit Theorem* says that, as n goes to infinity

$$P(a \leq T_n \leq b) \;\rightarrow\; \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

In other words, for n large, T_n is approximately a $N(0,1)$ random variable.

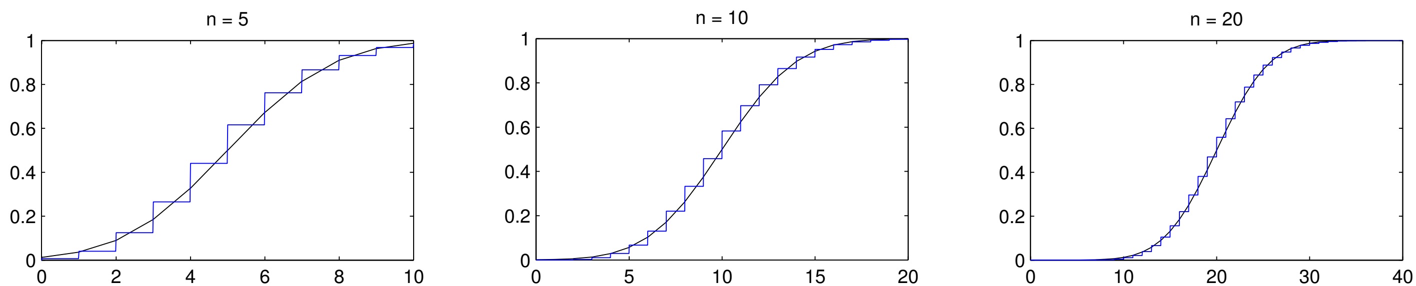
Poisson Random Variables

Let $X_1, X_2, \ldots$ be a sequence of independent Poisson random variables with parameter $\lambda = 1$ and let

$$S_n = \sum_{i=1}^{n} X_i$$

We saw before that the sum of independent Poisson random variables is again Poisson. Hence, S_n is Poisson with parameter $\lambda = n$. Hence, it has mean n and variance n.

By the Central Limit Theorem, the distribution should look like a Normal with the same mean and variance when n gets large.



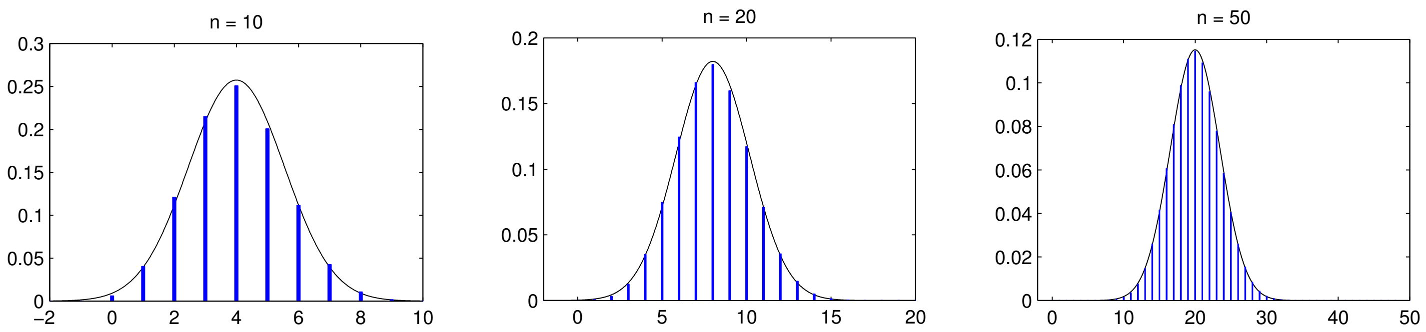

```
x = [(0:(2*n))-0.01 0:(2*n)];
x = sort(x);
P = cdf('poiss', x, n);
N = cdf('norm', x, n, sqrt(n));
plot(x,N,'k-'); hold on;
plot(x,P,'b-'); hold off;
```

Binomial Random Variables

Let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli rv's with parameter $p = 0.4$ and let

$$S_n = \sum_{i=1}^{n} X_i$$

The sum of n independent Bernoulli random variables is Binomial with parameters n and p.
Hence, it has mean np and variance npq, where $q = 1 - p$.
By the Central Limit Theorem, the distribution should look like a Normal with the same mean and variance when n gets large.




```
x = [-2:0.01:n];
N = pdf('norm', x, n*p, sqrt(n*p*q));
plot(x,N,'k-'); hold on;
x = [-2:n];
B = pdf('bino', x, n, p);
bar(x,B,0.1,'FaceColor','b','EdgeColor','b'); hold off;
```

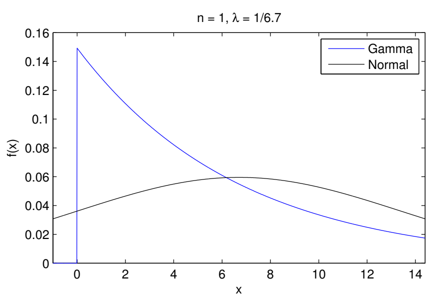
Gamma Random Variables

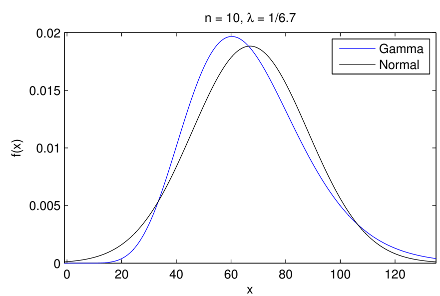
Let $X_1, X_2, \ldots$ be a sequence of independent exponential random variables with parameter λ and let

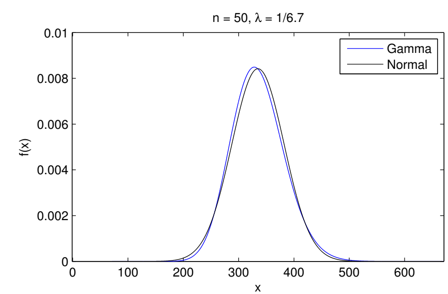
$$S_n = \sum_{i=1}^{n} X_i$$

We saw before that the sum of n independent exponential random variables is a Gamma random variable with parameters n and λ. Hence, it has mean n/λ and variance n/λ^2.

By the Central Limit Theorem, the distribution should look like a Normal with the same mean and variance when n gets large.



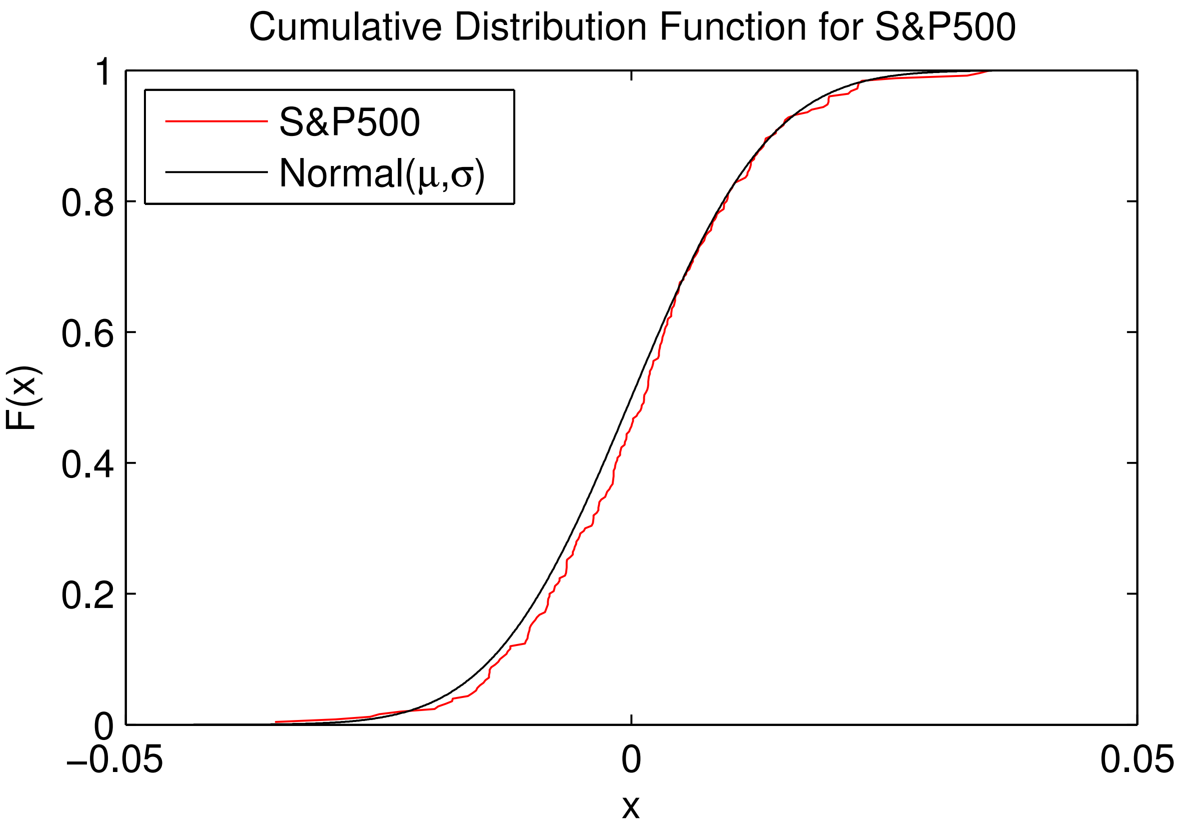





```
x = [-1:0.01:1+(2*n/lambda)];
G = pdf('gam', x, n, 1/lambda);
N = pdf('norm', x, n/lambda, sqrt(n)/lambda);
plot(x,G,'b-'); hold on;
plot(x,N,'k-'); hold off;
```

Recall: Standard and Poors 500 – Daily Returns

$$\mu = \mathbb{E}(R_i) \approx \sum_j R_j/n = 9.86 \times 10^{-4}, \quad \sigma^2 = \mathsf{Var}(R_i) \approx \sum_j (R_j - \mu)^2/n = 0.0108$$



Why the excellent match? *The CENTRAL LIMIT THEOREM!*