# ORF 245 Fundamentals of Statistics
# Chapter 6
# $\chi^2$, $t$, and $F$ Distributions

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# $\chi^2$ Distribution

Let $X_1, X_2, \ldots, X_n$ be a sequence of iid standard normal random variables and let

$$Z = \sum_{i=1}^{n} X_i^2$$

The random variable $Z$ is said to have a $\chi_n^2$ distribution.
For $n = 1$, the distribution is easy to compute explicitly. We begin with the cdf:

$$\begin{aligned} P(Z \leq z) &= P(X^2 \leq z) = P(-\sqrt{z} \leq X \leq \sqrt{z}) \\ &= \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 2 \int_0^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \end{aligned}$$

Differentiating, we get the density function:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} z^{-1/2} e^{-z/2}$$

We omit the derivation, but for general values of $n$ the density has the form:

$$f_Z(z) \approx z^{n/2-1} e^{-z/2}, \qquad z \geq 0$$

Note: This is a Gamma distribution with parameters "$n$" $= n/2$ and $\lambda = 1/2$.

# Student's $t$-Distribution

Let $Z \sim N(0,1)$ and let $U \sim \chi^2_n$.

Suppose that $Z$ and $U$ are independent.

The distribution of

$$T = \frac{Z}{\sqrt{U/n}}$$

is called the *$t$ distribution* with $n$ *degrees of freedom*.

We omit the derivation, but the density function has an "explicit" formula

$$f_T(t) = c\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

where $c$ is a normalization constant to make the area under the density equal to one.

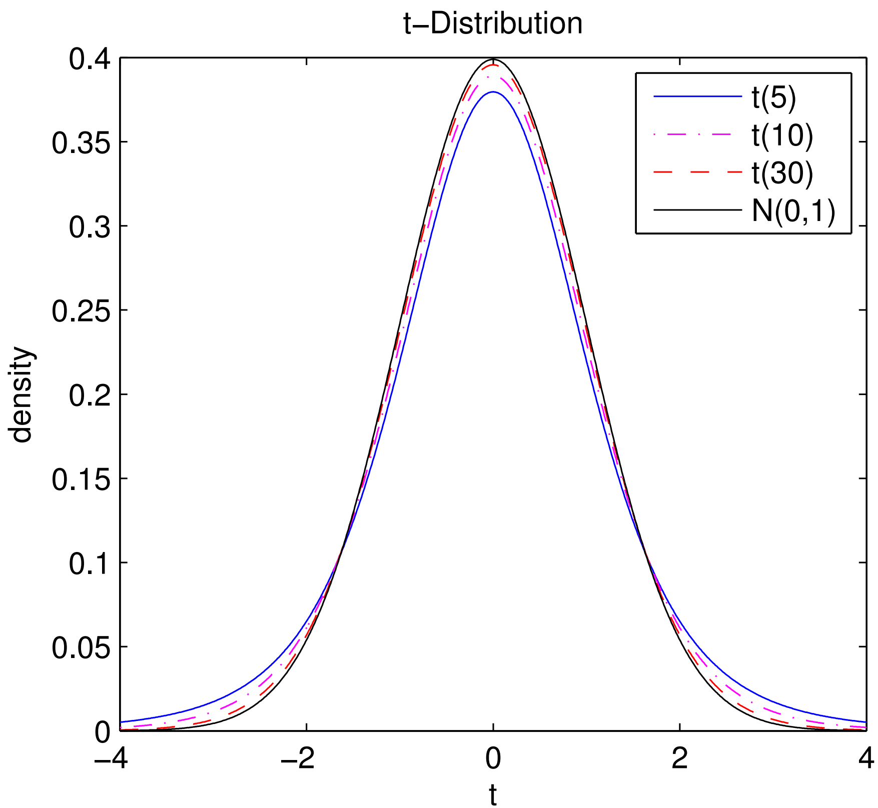
Note: When $n = 1$, this is the Cauchy distribution.

# The $t$ Distribution vs. $N(0,1)$

```
n = 10;
t = -4:0.01:4 ;
c = gamma((n+1)/2);
c = c / (sqrt(n*pi) * gamma(n/2) );
f = c*(1+t.^2/n).^(-(n+1)/2);
plot(t,f,'g--');
```

Alternatively...

```
n = 10;
t = -4:0.01:4 ;
f = pdf('t',t,n);
plot(t,f,'g--');
```




For $n \geq 30$ or so, the $t$-distribution is very well approximated by the standard normal distribution.

# The $F$ Distribution

Let $U \sim \chi^2_m$ and let $V \sim \chi^2_n$.

The distribution of

$$W = \frac{U/m}{V/n}$$

is called the $F$ *distribution* with $m$ and $n$ degrees of freedom and is denoted by $F_{m,n}$.

We omit the derivation, but the density function has an "explicit" formula

$$f_W(w) = c\, w^{m/2-1} \left(1 + \frac{m}{n} w\right)^{-(m+n)/2}, \qquad w \geq 0$$

where $c$ is a normalization constant to make the area under the density equal to one.

# Sample Mean and Sample Variance

Let $X_1, X_2, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables.

Such a collection is often called a *sample* from the distribution.

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The *sample mean* is defined as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Recall that

$$\mathbb{E}(\bar{X}) = \mu \qquad \text{and} \qquad \mathsf{Var}(\bar{X}) = \sigma^2/n$$

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The *sample variance* is defined as

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Why $n-1$ in the denominator?

# Biased vs. Unbiased Estimator

We compute:

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n}(X_i-\bar{X})^2\right) &= \sum_{i=1}^{n}\mathbb{E}(X_i^2-2X_i\bar{X}+\bar{X}^2)\\
&= n\mathbb{E}(X^2)-2\sum_{i,j}\mathbb{E}\left(\frac{X_iX_j}{n}\right)+n\sum_{j,k}\mathbb{E}\left(\frac{X_jX_k}{n^2}\right)\\
&= n\mathbb{E}(X^2)-\frac{1}{n}\sum_{i,j}\mathbb{E}\left(X_iX_j\right)\\
&= n\mathbb{E}(X^2)-\frac{1}{n}\sum_{i\neq j}\mathbb{E}\left(X_iX_j\right)-\frac{1}{n}\sum_{i=j}\mathbb{E}\left(X_iX_j\right)\\
&= n\mathbb{E}(X^2)-n(n-1)\frac{\mu^2}{n}-\mathbb{E}(X^2)\\
&= (n-1)\left(\mathbb{E}(X^2)-\mu^2\right)\\
&= (n-1)\sigma^2
\end{aligned}$$

Hence, $\mathbb{E}(S^2)=\sigma^2.$

# Three Results for Later

## Corollary A

The random variables $\bar{X}$ and $S^2$ are *independent*.

## Theorem B

The random variable $(n-1)S^2/\sigma^2$ is *chi-square* with $n-1$ degrees of freedom.

## Corollary B

The random variable

$$\frac{\bar{X}-\mu}{S/\sqrt{n}}$$

has a *t-distribution* with $n-1$ degrees of freedom.