simply infeasible, as the following example illustrates:

$$
\begin{array}{lrcr}
\text{maximize} & 5x_1 + 4x_2 & & \\
\text{subject to} & x_1 + \;\; x_2 & \leq & 2 \\
& -2x_1 - 2x_2 & \leq & -9 \\
& x_1,\; x_2 & \geq & 0.
\end{array}
$$

Indeed, the second constraint implies that $x_1 + x_2 \geq 4.5$, which contradicts the first constraint. If a problem has no feasible solution, then the problem itself is called *infeasible*.

At the other extreme from infeasible problems, one finds unbounded problems. A problem is *unbounded* if it has feasible solutions with arbitrarily large objective values. For example, consider

$$
\begin{array}{lrcr}
\text{maximize} & x_1 - 4x_2 & & \\
\text{subject to} & -2x_1 + \;\; x_2 & \leq & -1 \\
& -x_1 - 2x_2 & \leq & -2 \\
& x_1,\; x_2 & \geq & 0.
\end{array}
$$

Here, we could set x_2 to zero and let x_1 be arbitrarily large. As long as x_1 is greater than 2 the solution will be feasible, and as it gets large the objective function does too. Hence, the problem is unbounded. In addition to finding optimal solutions to linear programming problems, we shall also be interested in detecting when a problem is infeasible or unbounded.

Exercises

1.1 A steel company must decide how to allocate next week's time on a rolling mill, which is a machine that takes unfinished slabs of steel as input and can produce either of two semi-finished products: bands and coils. The mill's two products come off the rolling line at different rates:

Bands	200 tons/hr
Coils	140 tons/hr .

They also produce different profits:

Bands	\$ 25/ton
Coils	\$ 30/ton .

Based on currently booked orders, the following upper bounds are placed on the amount of each product to produce:

Bands	6000 tons
Coils	4000 tons .

Given that there are 40 hours of production time available this week, the problem is to decide how many tons of bands and how many tons of coils should be produced to yield the greatest profit. Formulate this problem as a linear programming problem. Can you solve this problem by inspection?

1.2 A small airline, Ivy Air, flies between three cities: Ithaca, Newark, and Boston. They offer several flights but, for this problem, let us focus on the Friday afternoon flight that departs from Ithaca, stops in Newark, and continues to Boston. There are three types of passengers:

(a) Those traveling from Ithaca to Newark.
(b) Those traveling from Newark to Boston.
(c) Those traveling from Ithaca to Boston.

The aircraft is a small commuter plane that seats 30 passengers. The airline offers three fare classes:

(a) Y class: full coach.
(b) B class: nonrefundable.
(c) M class: nonrefundable, 3-week advanced purchase.

Ticket prices, which are largely determined by external influences (i.e., competitors), have been set and advertised as follows:

	Ithaca–Newark	Newark–Boston	Ithaca–Boston
Y	300	160	360
B	220	130	280
M	100	80	140

Based on past experience, demand forecasters at Ivy Air have determined the following upper bounds on the number of potential customers in each of the 9 possible origin-destination/fare-class combinations:

	Ithaca–Newark	Newark–Boston	Ithaca–Boston
Y	4	8	3
B	8	13	10
M	22	20	18

The goal is to decide how many tickets from each of the 9 origin/destination/fare-class combinations to sell. The constraints are that the plane cannot be overbooked on either of the two legs of the flight and that the number of tickets made available cannot exceed the forecasted maximum demand. The objective is to maximize the revenue. Formulate this problem as a linear programming problem.

1.3 Suppose that Y is a random variable taking on one of n known values:

$$a_1, a_2, \ldots, a_n.$$

Suppose we know that Y either has distribution p given by

$$\mathbb{P}(Y = a_j) = p_j$$

or it has distribution q given by

$$\mathbb{P}(Y = a_j) = q_j.$$

Of course, the numbers $p_j,\ j = 1, 2, \ldots, n$ are nonnegative and sum to one. The same is true for the q_j's. Based on a single observation of Y, we wish to guess whether it has distribution p or distribution q. That is, for each possible outcome a_j, we will assert with probability x_j that the distribution is p and with probability $1-x_j$ that the distribution is q. We wish to determine the probabilities $x_j,\ j = 1, 2, \ldots, n$, such that the probability of saying the distribution is p when in fact it is q has probability no larger than β, where β is some small positive value (such as 0.05). Furthermore, given this constraint, we wish to maximize the probability that we say the distribution is p when in fact it is p. Formulate this maximization problem as a linear programming problem.

Notes

The subject of linear programming has its roots in the study of linear inequalities, which can be traced as far back as 1826 to the work of Fourier. Since then, many mathematicians have proved special cases of the most important result in the subject—the *duality theorem*. The applied side of the subject got its start in 1939 when L.V. Kantorovich noted the practical importance of a certain class of linear programming problems and gave an algorithm for their solution—see Kantorovich (1960). Unfortunately, for several years, Kantorovich's work was unknown in the West and unnoticed in the East. The subject really took off in 1947 when G.B. Dantzig invented the *simplex method* for solving the linear programming problems that arose in U.S. Air Force planning problems. The earliest published accounts of Dantzig's work appeared in 1951 (Dantzig 1951*a*,*b*). His monograph (Dantzig 1963) remains an important reference. In the same year that Dantzig invented the simplex method, T.C. Koopmans showed that linear programming provided the appropriate model for the analysis of classical economic theories. In 1975, the Royal Swedish Academy of Sciences awarded the Nobel Prize in economic science to L.V. Kantorovich and T.C. Koopmans "for their contributions to the theory of optimum allocation of resources." Apparently the academy regarded Dantzig's work as too mathematical for the prize in economics (and there is no Nobel Prize in mathematics).