At this point, no further reductions are possible. Computing the payoff matrix, we get

$$
A = \begin{array}{c} \\ (1,1,2) \\ (1,1,3) \\ (1,2,2) \\ (1,2,3) \\ (3,1,2) \\ (3,1,3) \\ (3,2,2) \\ (3,2,3) \end{array}
\begin{array}{c}
\begin{array}{cccc} (1,1,4) & (1,2,4) & (3,1,4) & (3,2,4) \end{array} \\
\left[\begin{array}{cccc}
 & & \frac{1}{6} & \frac{1}{6} \\
 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\
\frac{1}{6} & & & -\frac{1}{6} \\
-\frac{1}{6} & \frac{1}{3} & & \frac{1}{2} \\
-\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\
 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \\
 & \frac{1}{3} & -\frac{1}{6} & \frac{1}{6}
\end{array}\right]
\end{array}.
$$

Solving the matrix game, we find that

$$
y^* = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix}^T
$$

and

$$
x^* = \begin{bmatrix} \frac{2}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix}^T.
$$

These stochastic vectors can be summarized as simple statements of the optimal randomized strategies for the two players. Indeed, player A's optimal strategy is as follows:

when holding 1, mix lines 1 and 3 in 5:1 proportion;
when holding 2, mix lines 1 and 2 in 1:1 proportion;
when holding 3, mix lines 2 and 3 in 1:1 proportion.

Similarly, player B's optimal strategy can be described as

when holding 1, mix lines 1 and 3 in 2:1 proportion;
when holding 2, mix lines 1 and 2 in 2:1 proportion;
when holding 3, use line 4.

Note that it is optimal for player A to use line 3 when holding a 1 at least some of the time. Since line 3 says to bet, this bet is a bluff. Player B also bluffs sometimes, since betting line 3 is sometimes used when holding a 1. Clearly, the optimal strategies also exhibit some underbidding.

Exercises

11.1 Players A and B each hide a nickel or a dime. If the hidden coins match, player A gets both; if they don't match, then B gets both. Find the optimal strategies. Which player has the advantage? Solve the problem for arbitrary denominations a and b.

11.2 Players A and B each pick a number between 1 and 100. The game is a draw if both players pick the same number. Otherwise, the player who picks the smaller number wins unless that smaller number is one less than the opponent's number, in which case the opponent wins. Find the optimal strategy for this game.

11.3 We say that row r *dominates* row s if $a_{rj} \geq a_{sj}$ for all $j = 1, 2, \ldots, n$. Similarly, column r is said to dominate column s if $a_{ir} \geq a_{is}$ for all $i = 1, 2, \ldots, m$. Show that

(a) If a row (say, r) dominates another row, then the row player has an optimal strategy y^* in which $y_r^* = 0$.

(b) If a column (say, s) is dominated by another column, then the column player has an optimal strategy x^* in which $x_s^* = 0$.

Use these results to reduce the following payoff matrix to a 2×2 matrix:

$$\begin{bmatrix} -6 & 2 & -4 & -7 & -5 \\ 0 & 4 & -2 & -9 & -1 \\ -7 & 3 & -3 & -8 & -2 \\ 2 & -3 & 6 & 0 & 3 \end{bmatrix}.$$

11.4 Solve simplified poker assuming that antes are \$2 and bets are \$1.

11.5 Give necessary and sufficient conditions for the rth pure strategy of the row and the sth pure strategy of the column player to be simultaneously optimal.

11.6 Use the Minimax Theorem to show that

$$\max_x \min_y \; y^T A x = \min_y \max_x \; y^T A x.$$

11.7 *Bimatrix Games*. Consider the following two-person game defined in terms of a pair of $m \times n$ matrices A and B: if the row player selects row index i and the column player selects column index j, then the row player pays a_{ij} dollars and the column player pays b_{ij} dollars. Stochastic vectors x^* and y^* are said to form a *Nash equilibrium* if

$$\begin{aligned} {y^*}^T A x^* &\leq y^T A x^* \qquad \text{for all } y \\ {y^*}^T B x^* &\leq {y^*}^T B x \qquad \text{for all } x. \end{aligned}$$

The purpose of this exercise is to relate Nash equilibria to the problem of finding vectors x and y that satisfy

$$
\begin{aligned}
\begin{bmatrix} 0 & -A \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix} &= \begin{bmatrix} -e \\ -e \end{bmatrix}, \\
y_i w_i &= 0, \qquad \text{for all } i, \\
x_j z_j &= 0, \qquad \text{for all } j, \\
x, w, y, z &\geq 0
\end{aligned}
\tag{11.5}
$$

(vectors w and z can be thought of as being defined by the matrix equality). Problem (11.5) is called a *linear complementarity problem*.

(a) Show that there is no loss in generality in assuming that A and B have all positive entries.

(b) Assuming that A and B have all positive entries, show that, if (x^*, y^*) is a Nash equilibrium, then

$$
x' = \frac{x^*}{y^{*T} A x^*}, \qquad y' = \frac{y^*}{y^{*T} B x^*}
$$

solves the linear complementarity problem (11.5).

(c) Show that, if (x', y') solves the linear complementarity problem (11.5), then

$$
x^* = \frac{x'}{e^T x'}, \qquad y^* = \frac{y'}{e^T y'}
$$

is a Nash equilibrium.

(An algorithm for solving the linear complementarity problem is developed in Exercise 18.7.)

11.8 *The Game of Morra.* Two players simultaneously throw out one or two fingers and call out their guess as to what the total sum of the outstretched fingers will be. If a player guesses right, but his opponent does not, he receives payment equal to his guess. In all other cases, it is a draw.

(a) List the pure strategies for this game.

(b) Write down the payoff matrix for this game.

(c) Formulate the row player's problem as a linear programming problem. (*Hint: Recall that the row player's problem is to minimize the maximum expected payout*.)

(d) What is the value of this game?

(e) Find the optimal randomized strategy.

11.9 *Heads I Win—Tails You Lose.* In the classical coin-tossing game, player A tosses a fair coin. If it comes up heads player B pays player A \$2 but if it comes up tails player A pays player B \$2. As a two-person zero-sum game, this game is rather trivial since neither player has anything to *decide* (after agreeing to play the game). In fact, the matrix for this game is a 1×1 matrix with only a zero in it, which represents the expected payoff from player A to B.

Now consider the same game with the following twist. Player A is allowed to peek at the outcome and then decide either to stay in the game or to bow out. If player A bows out, then he automatically loses but only has to pay player B \$1. Of course, player A must inform player B of his decision. If his decision is to stay in the game, then player B has the option either to stay in the game or not. If she decides to get out, then she loses \$1 to player A. If both players stay in the game, then the rules are as in the classical game: heads means player A wins, tails means player B wins.

(a) List the strategies for each player in this game. (Hint: Don't forget that a strategy is something that a player has control over.)
(b) Write down the payoff matrix.
(c) A few of player A's strategies are uniformly inferior to others. These strategies can be ruled out. Which of player A's strategies can be ruled out?
(d) Formulate the row player's problem as a linear programming problem. (*Hints: (1) Recall that the row player's problem is to minimize the maximum expected payout. (2) Don't include rows that you ruled out in the previous part*.)
(e) Find the optimal randomized strategy.
(f) Discuss whether this game is interesting or not.

Notes

The Minimax Theorem was proved by von Neumann (1928). Important references include Gale et al. (1951), von Neumann & Morgenstern (1947), Karlin (1959), and Dresher (1961). Simplified poker was invented and analyzed by Kuhn (1950). Exercises 11.1 and 11.2 are borrowed from Chvátal (1983).