Put
\[ \hat{f} = f(\bar{x}, \bar{w}) \]
and let
\[ \bar{P} = \{(x, w) : Ax + w = b, x \geq 0, w \geq 0, f(x, w) \geq \bar{f}\}. \]
Clearly, \( \bar{P} \) is nonempty, since it contains \((\bar{x}, \bar{w})\). From the discussion above, we see that \( \bar{P} \) is a bounded set.

This set is also closed. To see this, note that it is the intersection of three sets,
\[ \{(x, w) : Ax + w = b\} \cap \{(x, w) : x \geq 0, w \geq 0\} \cap \{(x, w) : f(x, w) \geq \bar{f}\}. \]
The first two of these sets are obviously closed. The third set is closed because it is the inverse image of a closed set, \([\bar{f}, \infty]\), under a continuous mapping \(f\). Finally, the intersection of three closed sets is closed.

In Euclidean spaces, a closed bounded set is called compact. A well-known theorem from real analysis about compact sets is that a continuous function on a nonempty compact set attains its maximum. This means that there exists a point in the compact set at which the function hits its maximum. Applying this theorem to \(f\) on \(\bar{P}\), we see that \(f\) does indeed attain its maximum on \(\bar{P}\), and this implies it attains its maximum on all of \{(x, w) : x > 0, w > 0\}, since \(\bar{P}\) was by definition that part of this domain on which \(f\) takes large values (bigger than \(\bar{f}\), anyway). This completes the proof. \(\square\)

We summarize our main result in the following corollary:

**Corollary 17.3.** If a primal feasible set (or, for that matter, its dual) has a nonempty interior and is bounded, then for each \(\mu > 0\) there exists a unique solution \((x_\mu, w_\mu, y_\mu, z_\mu)\) to (17.6).

**Proof.** Follows immediately from the previous theorem and Exercise 10.7 \(\square\)

The path \(\{(x_\mu, w_\mu, y_\mu, z_\mu) : \mu > 0\}\) is called the primal–dual central path. It plays a fundamental role in interior-point methods for linear programming. In the next chapter, we define the simplest interior-point method. It is an iterative procedure that at each iteration attempts to move toward a point on the central path that is closer to optimality than the current point.

**Exercises**

17.1 Compute and graph the central trajectory for the following problem:

\[
\begin{align*}
\text{maximize} & \quad -x_1 + x_2 \\
\text{subject to} & \quad x_2 \leq 1 \\
& \quad -x_1 \leq -1 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
**Hint:** The primal and dual problems are the same — exploit this symmetry.

**17.2** Let $\theta$ be a fixed parameter, $0 \leq \theta \leq \frac{\pi}{2}$, and consider the following problem:

$$
\text{maximize} \quad (\cos \theta)x_1 + (\sin \theta)x_2 \\
\text{subject to} \quad x_1 \leq 1 \\
\quad \quad \quad \quad \quad x_2 \leq 1 \\
\quad \quad \quad \quad \quad x_1, x_2 \geq 0.
$$

Compute an explicit formula for the central path $(x_\mu, w_\mu, y_\mu, z_\mu)$, and evaluate $\lim_{\mu \to \infty} x_\mu$ and $\lim_{\mu \to 0} x_\mu$.

**17.3** Suppose that \{x : Ax \leq b, x \geq 0\} is bounded. Let $r \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$ be vectors with positive elements. By studying an appropriate barrier function, show that there exists a unique solution to the following nonlinear system:

$$
Ax + w = b \\
ATy - z = c \\
XZe = r \\
YWc = s \\
x, y, z, w > 0.
$$

**17.4** Consider the linear programming problem in equality form:

$$
\text{(17.8) maximize} \quad \sum_j c_j x_j \\
\text{subject to} \quad \sum_j a_j x_j = b \\
\quad \quad \quad \quad x_j \geq 0, \quad j = 1, 2, \ldots, n,
$$

where each $a_j$ is a vector in $\mathbb{R}^m$, as is $b$. Consider the change of variables,

$$
x_j = \xi_j^2,
$$

and the associated maximization problem:

$$
\text{(17.9) maximize} \quad \sum_j c_j \xi_j^2 \\
\text{subject to} \quad \sum_j a_j \xi_j^2 = b
$$

(note that the nonnegativity constraints are no longer needed). Let $V$ denote the set of basic feasible solutions to (17.8), and let $W$ denote the set of points $(\xi_1^2, \xi_2^2, \ldots, \xi_n^2)$ in $\mathbb{R}^n$ for which $(\xi_1, \xi_2, \ldots, \xi_n)$ is a solution to the first-order optimality conditions for (17.9). Show that $V \subset W$. What does this say about the possibility of using (17.9) as a vehicle to solve (17.8)?