Put

$$\bar{f} = f(\bar{x}, \bar{w})$$

and let

$$\bar{P} = \{(x, w) : Ax + w = b, x \geq 0, w \geq 0, f(x, w) \geq \bar{f}\}.$$

Clearly, $\bar{P}$ is nonempty, since it contains $(\bar{x}, \bar{w})$. From the discussion above, we see that $\bar{P}$ is a bounded set.

This set is also closed. To see this, note that it is the intersection of three sets,

$$\{(x, w) : Ax + w = b\} \cap \{(x, w) : x \geq 0, w \geq 0\} \cap \{(x, w) : f(x, w) \geq \bar{f}\}.$$

The first two of these sets are obviously closed. The third set is closed because it is the inverse image of a closed set, $[\bar{f}, \infty]$, under a continuous mapping f. Finally, the intersection of three closed sets is closed.

In Euclidean spaces, a closed bounded set is called compact. A well-known theorem from real analysis about compact sets is that a continuous function on a nonempty compact set attains its maximum. This means that there exists a point in the compact set at which the function hits its maximum. Applying this theorem to f on $\bar{P}$, we see that f does indeed attain its maximum on $\bar{P}$, and this implies it attains its maximum on all of $\{(x, w) : x > 0, w > 0\}$, since $\bar{P}$ was by definition that part of this domain on which f takes large values (bigger than $\bar{f}$, anyway). This completes the proof. □

We summarize our main result in the following corollary:

COROLLARY 17.3. *If a primal feasible set (or, for that matter, its dual) has a nonempty interior and is bounded, then for each* $\mu > 0$ *there exists a unique solution*

$$(x_\mu, w_\mu, y_\mu, z_\mu)$$

to (17.6).

PROOF. Follows immediately from the previous theorem and Exercise 10.7. □

The path $\{(x_\mu, w_\mu, y_\mu, z_\mu) : \mu > 0\}$ is called the *primal–dual central path*. It plays a fundamental role in interior-point methods for linear programming. In the next chapter, we define the simplest interior-point method. It is an iterative procedure that at each iteration attempts to move toward a point on the central path that is closer to optimality than the current point.

Exercises

17.1 Compute and graph the central trajectory for the following problem:

$$\begin{array}{lrcr} \text{maximize} & -x_1 + x_2 & & \\ \text{subject to} & x_2 & \leq & 1 \\ & -x_1 \quad & \leq & -1 \\ & x_1,\ x_2 & \geq & 0. \end{array}$$

Hint: The primal and dual problems are the same — exploit this symmetry.

17.2 Let θ be a fixed parameter, $0 \leq \theta \leq \frac{\pi}{2}$, and consider the following problem:

$$\begin{array}{ll} \text{maximize} & (\cos\theta)x_1 + (\sin\theta)x_2 \\ \text{subject to} & x_1 \leq 1 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0. \end{array}$$

Compute an explicit formula for the central path $(x_\mu, w_\mu, y_\mu, z_\mu)$, and evaluate $\lim_{\mu\to\infty} x_\mu$ and $\lim_{\mu\to 0} x_\mu$.

17.3 Suppose that $\{x : Ax \leq b, x \geq 0\}$ is bounded. Let $r \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$ be vectors with positive elements. By studying an appropriate barrier function, show that there exists a unique solution to the following nonlinear system:

$$\begin{aligned} Ax + w &= b \\ A^T y - z &= c \\ XZe &= r \\ YWe &= s \\ x, y, z, w &> 0. \end{aligned}$$

17.4 Consider the linear programming problem in equality form:

$$(17.8) \qquad \begin{array}{ll} \text{maximize} & \sum_j c_j x_j \\ \text{subject to} & \displaystyle\sum_j a_j x_j = b \\ & x_j \geq 0, \qquad j = 1, 2, \ldots, n, \end{array}$$

where each a_j is a vector in $\mathbb{R}^m$, as is b. Consider the change of variables,

$$x_j = \xi_j^2,$$

and the associated maximization problem:

$$(17.9) \qquad \begin{array}{ll} \text{maximize} & \sum_j c_j \xi_j^2 \\ \text{subject to} & \sum_j a_j \xi_j^2 = b \end{array}$$

(note that the nonnegativity constraints are no longer needed). Let V denote the set of basic feasible solutions to (17.8), and let W denote the set of points $(\xi_1^2, \xi_2^2, \ldots, \xi_n^2)$ in $\mathbb{R}^n$ for which $(\xi_1, \xi_2, \ldots, \xi_n)$ is a solution to the first-order optimality conditions for (17.9). Show that $V \subset W$. What does this say about the possibility of using (17.9) as a vehicle to solve (17.8)?