Clearly, this is the maximum value of the objective function. The optimal solution is the intersection of this level set with the set of feasible solutions. Hence, we see from Figure 2.1 that the optimal solution is $(x_1, x_2) = (6, 2)$.

Exercises

Solve the following linear programming problems. If you wish, you may check your arithmetic by using the simple online pivot tool:

www.princeton.edu/∼rvdb/JAVA/pivot/simple.html

2.1
$$\begin{array}{ll} \text{maximize} & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{subject to} & 2x_1 + x_2 + x_3 + 3x_4 \le 5 \\ & x_1 + 3x_2 + x_3 + 2x_4 \le 3 \\ & x_1, \; x_2, \; x_3, \; x_4 \ge 0. \end{array}$$

2.2
$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \le 4 \\ & 2x_1 + 3x_2 \le 3 \\ & 4x_1 + x_2 \le 5 \\ & x_1 + 5x_2 \le 1 \\ & x_1, \; x_2 \ge 0. \end{array}$$

2.3
$$\begin{array}{ll} \text{maximize} & 2x_1 - 6x_2 \\ \text{subject to} & -x_1 - x_2 - x_3 \le -2 \\ & 2x_1 - x_2 + x_3 \le 1 \\ & x_1, \; x_2, \; x_3 \ge 0. \end{array}$$

2.4
$$\begin{array}{ll} \text{maximize} & -x_1 - 3x_2 - x_3 \\ \text{subject to} & 2x_1 - 5x_2 + x_3 \le -5 \\ & 2x_1 - x_2 + 2x_3 \le 4 \\ & x_1, \; x_2, \; x_3 \ge 0. \end{array}$$

2.5
$$\begin{array}{ll} \text{maximize} & x_1 + 3x_2 \\ \text{subject to} & -x_1 - x_2 \le -3 \\ & -x_1 + x_2 \le -1 \\ & x_1 + 2x_2 \le 4 \\ & x_1, \; x_2 \ge 0. \end{array}$$

2.6
$$\begin{array}{ll} \text{maximize} & x_1 + 3x_2 \\ \text{subject to} & -x_1 - x_2 \le -3 \\ & -x_1 + x_2 \le -1 \\ & x_1 + 2x_2 \le 2 \\ & x_1,\ x_2 \ge 0. \end{array}$$

2.7
$$\begin{array}{ll} \text{maximize} & x_1 + 3x_2 \\ \text{subject to} & -x_1 - x_2 \le -3 \\ & -x_1 + x_2 \le -1 \\ & -x_1 + 2x_2 \le 2 \\ & x_1,\ x_2 \ge 0. \end{array}$$

2.8
$$\begin{array}{ll} \text{maximize} & 3x_1 + 2x_2 \\ \text{subject to} & x_1 - 2x_2 \le 1 \\ & x_1 - x_2 \le 2 \\ & 2x_1 - x_2 \le 6 \\ & x_1 \le 5 \\ & 2x_1 + x_2 \le 16 \\ & x_1 + x_2 \le 12 \\ & x_1 + 2x_2 \le 21 \\ & x_2 \le 10 \\ & x_1,\ x_2 \ge 0. \end{array}$$

2.9
$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 + 4x_3 \\ \text{subject to} & -2x_2 - 3x_3 \ge -5 \\ & x_1 + x_2 + 2x_3 \le 4 \\ & x_1 + 2x_2 + 3x_3 \le 7 \\ & x_1,\ x_2,\ x_3 \ge 0. \end{array}$$

2.10
$$\begin{array}{ll} \text{maximize} & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1,\ x_2,\ x_3,\ x_4 \ge 0. \end{array}$$

2.11

$$
\begin{array}{lrcl}
\text{minimize} & x_{12} + 8x_{13} + 9x_{14} + 2x_{23} + 7x_{24} + 3x_{34} & & \\
\text{subject to} & x_{12} + x_{13} + x_{14} & \geq & 1 \\
 & -x_{12} + x_{23} + x_{24} & = & 0 \\
 & -x_{13} - x_{23} + x_{34} & = & 0 \\
 & x_{14} + x_{24} + x_{34} & \leq & 1 \\
 & x_{12}, x_{13}, \ldots, x_{34} & \geq & 0.
\end{array}
$$

2.12 Using today's date (MMYY) for the seed value, solve 10 initially feasible problems using the online pivot tool:

www.princeton.edu/∼rvdb/JAVA/pivot/primal.html

2.13 Using today's date (MMYY) for the seed value, solve 10 not necessarily feasible problems using the online pivot tool:

www.princeton.edu/∼rvdb/JAVA/pivot/primal_x0.html

2.14 Consider the following dictionary:

$$
\begin{array}{rcl}
\zeta & = & 3 + x_1 + 6x_2 \\
\hline
w_1 & = & 1 + x_1 - x_2 \\
w_2 & = & 5 - 2x_1 - 3x_2.
\end{array}
$$

Using the largest coefficient rule to pick the entering variable, compute the dictionary that results after *one pivot*.

2.15 Show that the following dictionary cannot be the optimal dictionary for any linear programming problem in which w_1 and w_2 are the initial slack variables:

$$
\begin{array}{rcl}
\zeta & = & 4 - w_1 - 2x_2 \\
\hline
x_1 & = & 3 \qquad - 2x_2 \\
w_2 & = & 1 + w_1 - x_2.
\end{array}
$$

Hint: if it could, what was the original problem from whence it came?

2.16 Graph the feasible region for Exercise 2.8. Indicate on the graph the sequence of basic solutions produced by the simplex method.

2.17 Give an example showing that the variable that becomes basic in one iteration of the simplex method can become nonbasic in the next iteration.

2.18 Show that the variable that becomes nonbasic in one iteration of the simplex method cannot become basic in the next iteration.

2.19 Solve the following linear programming problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} p_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} q_j x_j \leq \beta \\ & x_j \leq 1 \qquad j = 1, 2, \ldots, n \\ & x_j \geq 0 \qquad j = 1, 2, \ldots, n. \end{aligned}$$

Here, the numbers p_j, $j = 1, 2, \ldots, n$, are positive and sum to one. The same is true of the q_j's:

$$\sum_{j=1}^{n} q_j = 1$$
$$q_j > 0.$$

Furthermore (with only minor loss of generality), you may assume that

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n}.$$

Finally, the parameter β is a small positive number. See Exercise 1.3 for the motivation for this problem.

Notes

The simplex method was invented by G.B. Dantzig in 1949. His monograph (Dantzig 1963) is the classical reference. Most texts describe the simplex method as a sequence of pivots on a table of numbers called the *simplex tableau*. Following Chvátal (1983), we have developed the algorithm using the more memorable dictionary notation.