## Exercises

In solving the following problems, the advanced pivot tool can be used to check your arithmetic:

www.princeton.edu/∼rvdb/JAVA/pivot/advanced.html

**5.1** What is the dual of the following linear programming problem:

$$
\begin{array}{lrcl}
\text{maximize} & x_1 - 2x_2 & & \\
\text{subject to} & x_1 + 2x_2 - x_3 + x_4 & \geq & 0 \\
& 4x_1 + 3x_2 + 4x_3 - 2x_4 & \leq & 3 \\
& -x_1 - x_2 + 2x_3 + x_4 & = & 1 \\
& x_2,\ x_3 & \geq & 0 .
\end{array}
$$

**5.2** Illustrate Theorem 5.2 on the problem in Exercise 2.9.

**5.3** Illustrate Theorem 5.2 on the problem in Exercise 2.1.

**5.4** Illustrate Theorem 5.2 on the problem in Exercise 2.2.

**5.5** Consider the following linear programming problem:

$$
\begin{array}{lrcl}
\text{maximize} & 2x_1 + 8x_2 - x_3 - 2x_4 & & \\
\text{subject to} & 2x_1 + 3x_2 \qquad\quad + 6x_4 & \leq & 6 \\
& -2x_1 + 4x_2 + 3x_3 \qquad\quad & \leq & 1.5 \\
& 3x_1 + 2x_2 - 2x_3 - 4x_4 & \leq & 4 \\
& x_1,\ x_2,\ x_3,\ x_4 & \geq & 0.
\end{array}
$$

Suppose that, in solving this problem, you have arrived at the following dictionary:

$$
\begin{array}{rcl}
\zeta & = & 3.5 - 0.25w_1 + 6.25x_2 - 0.5w_3 - 1.5x_4 \\
\hline
x_1 & = & 3.0 - 0.5w_1 - 1.5x_2 \qquad\qquad\quad - 3.0x_4 \\
w_2 & = & 0.0 + 1.25w_1 - 3.25x_2 - 1.5w_3 + 13.5x_4 \\
x_3 & = & 2.5 - 0.75w_1 - 1.25x_2 + 0.5w_3 - 6.5x_4.
\end{array}
$$

(a) Write down the dual problem.
(b) In the dictionary shown above, which variables are basic? Which are nonbasic?
(c) Write down the primal solution corresponding to the given dictionary. Is it feasible? Is it degenerate?

(d) Write down the corresponding dual dictionary.
(e) Write down the dual solution. Is it feasible?
(f) Do the primal/dual solutions you wrote above satisfy the complementary slackness property?
(g) Is the current primal solution optimal?
(h) For the next (primal) pivot, which variable will enter if the largest coefficient rule is used? Which will leave? Will the pivot be degenerate?

**5.6** Solve the following linear program:

$$\begin{array}{lrcr} \text{maximize} & -x_1 - 2x_2 & & \\ \text{subject to} & -2x_1 + 7x_2 & \leq & 6 \\ & -3x_1 + \phantom{7}x_2 & \leq & -1 \\ & 9x_1 - 4x_2 & \leq & 6 \\ & x_1 - \phantom{7}x_2 & \leq & 1 \\ & 7x_1 - 3x_2 & \leq & 6 \\ & -5x_1 + 2x_2 & \leq & -3 \\ & x_1,\ x_2 & \geq & 0. \end{array}$$

**5.7** Solve the linear program given in Exercise 2.3 using the dual–primal two-phase algorithm.

**5.8** Solve the linear program given in Exercise 2.4 using the dual–primal two-phase algorithm.

**5.9** Solve the linear program given in Exercise 2.6 using the dual–primal two-phase algorithm.

**5.10** Using today's date (MMYY) for the seed value, solve 10 problems using the dual phase I primal phase II simplex method:

www.princeton.edu/∼rvdb/JAVA/pivot/dp2phase.html

**5.11** Using today's date (MMYY) for the seed value, solve 10 problems using the primal phase I dual phase II simplex method:

www.princeton.edu/∼rvdb/JAVA/pivot/pd2phase.html

**5.12** For $x$ and $y$ in $\mathbb{R}$, compute

$$\max_{x\geq 0}\ \min_{y\geq 0}\ (x-y) \quad \text{and} \quad \min_{y\geq 0}\ \max_{x\geq 0}\ (x-y)$$

and note whether or not they are equal.

**5.13** Consider the following process. Starting with a linear programming problem in standard form,

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, 2, \ldots, m \\ & x_j \geq 0 & j = 1, 2, \ldots, n, \end{array}$$

first form its dual:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} y_i a_{ij} \geq c_j & j = 1, 2, \ldots, n \\ & y_i \geq 0 & i = 1, 2, \ldots, m. \end{array}$$

Then replace the minimization in the dual with a maximization to get a new linear programming problem, which we can write in standard form as follows:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} -y_i a_{ij} \leq -c_j & j = 1, 2, \ldots, n \\ & y_i \geq 0 & i = 1, 2, \ldots, m. \end{array}$$

If we identify a linear programming problem with its data, $(a_{ij}, b_i, c_j)$, the above process can be thought of as a transformation $T$ on the space of data defined by

$$(a_{ij}, b_i, c_j) \xrightarrow{T} (-a_{ji}, -c_j, b_i).$$

Let $\zeta^*(a_{ij}, b_i, c_j)$ denote the optimal objective function value of the standard-form linear programming problem having data $(a_{ij}, b_i, c_j)$. By strong duality together with the fact that a maximization dominates a minimization, it follows that

$$\zeta^*(a_{ij}, b_i, c_j) \leq \zeta^*(-a_{ji}, -c_j, b_i).$$

Now if we repeat this process, we get

$$\begin{aligned}(a_{ij}, b_i, c_j) &\xrightarrow{T} (-a_{ji}, -c_j, b_i)\\ &\xrightarrow{T} (a_{ij}, -b_i, -c_j)\\ &\xrightarrow{T} (-a_{ji}, c_j, -b_i)\\ &\xrightarrow{T} (a_{ij}, b_i, c_j)\end{aligned}$$

and hence that

$$\begin{aligned}\zeta^*(a_{ij}, b_i, c_j) &\le \zeta^*(-a_{ji}, -c_j, b_i)\\ &\le \zeta^*(a_{ij}, -b_i, -c_j)\\ &\le \zeta^*(-a_{ji}, c_j, -b_i)\\ &\le \zeta^*(a_{ij}, b_i, c_j).\end{aligned}$$

But the first and the last entry in this chain of inequalities are equal. Therefore, all these inequalities would seem to be equalities. While this outcome could happen sometimes, it certainly isn't always true. What is the error in this logic? Can you state a (correct) nontrivial theorem that follows from this line of reasoning? Can you give an example where the four inequalities are indeed all equalities?

**5.14** Consider the following variant of the resource allocation problem:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \le b_i & i = 1, 2, \ldots, m \\ & 0 \le x_j \le u_j & j = 1, 2, \ldots, n. \end{array} \tag{5.17}$$

As usual, the $c_j$'s denote the unit prices for the products and the $b_i$'s denote the number of units on hand for each raw material. In this variant, the $u_j$'s denote upper bounds on the number of units of each product that can be sold at the set price. Now, let's assume that the raw materials have not been purchased yet and it is part of the problem to determine the $b_i$'s. Let $p_i$, $i = 1, 2, \ldots, m$ denote the price for raw material $i$. The problem then becomes an optimization over both the $x_j$'s and the $b_i$'s:

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} p_i b_i & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j - b_i \le 0 & i = 1, 2, \ldots, m \\
& 0 \le x_j \le u_j & j = 1, 2, \ldots, n \\
& b_i \ge 0 & i = 1, 2, \ldots, m.
\end{array}
$$

(a) Show that this problem always has an optimal solution.

(b) Let $y_i^*(b)$, $i = 1, 2, \ldots, m$, denote optimal dual variables for the original resource allocation problem (5.17). Note that we've explicitly indicated that these dual variables depend on the $b$'s. Also, we assume that problem (5.17) is both primal and dual non-degenerate so the $y_i^*(b)$ is uniquely defined. Show that the optimal value of the $b_i$'s, call them $b_i^*$'s, satisfy

$$y_i^*(b^*) = p_i.$$

*Hint: You will need to use the fact that, for resource allocation problems, we have $a_{ij} \ge 0$ for all $i$, and all $j$.*

**5.15** Consider the following linear program:

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=1}^{n} p_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} q_j x_j \le \beta & \\
& x_j \le 1 & j = 1, 2, \ldots, n \\
& x_j \ge 0 & j = 1, 2, \ldots, n.
\end{array}
$$

Here, the numbers $p_j$, $j = 1, 2, \ldots, n$ are positive and sum to one. The same is true of the $q_j$'s:

$$
\begin{aligned}
\sum_{j=1}^{n} q_j &= 1 \\
q_j &> 0.
\end{aligned}
$$

Furthermore, assume that

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n}$$

and that the parameter $\beta$ is a small positive number. Let $k = \min\{j : q_{j+1} + \cdots + q_n \le \beta\}$. Let $y_0$ denote the dual variable associated with the constraint involving $\beta$, and let $y_j$ denote the dual variable associated with the upper bound of 1 on variable $x_j$. Using duality theory, show that the optimal values of the primal and dual variables are given by

$$x_j = \begin{cases} 0 & j < k \\ \frac{\beta - q_{k+1} - \cdots - q_n}{q_k} & j = k \\ 1 & j > k \end{cases}$$

$$y_j = \begin{cases} \frac{p_k}{q_k} & j = 0 \\ 0 & 0 < j \leq k \\ q_j\left(\frac{p_j}{q_j} - \frac{p_k}{q_k}\right) & j > k \end{cases}$$

See Exercise 1.3 for the motivation for this problem. (Note: The set of indices defining the integer $k$ is never empty. To see this, note that for $j = n - 1$ the condition is $q_n \leq \beta$, which may or may not be true. But, for $j = n$, the sum on the left-hand side contains no terms and so the condition is $0 \leq \beta$, which is always true. Hence, the sum always contains at least one element... the number $n$.)

**5.16** *Diet Problem.* An MIT graduate student was trying to make ends meet on a very small stipend. He went to the library and looked up the National Research Council's publication entitled "Recommended Dietary Allowances" and was able to determine a minimum daily intake quantity of each essential nutrient for a male in his weight and age category. Let $m$ denote the number of nutrients that he identified as important to his diet, and let $b_i$ for $i = 1, 2, \ldots, m$ denote his personal minimum daily requirements. Next, he made a list of his favorite foods (which, except for pizza and due mostly to laziness and ineptitude in the kitchen, consisted almost entirely of frozen prepared meals). He then went to the local grocery store and made a list of the unit price for each of his favorite foods. Let us denote these prices as $c_j$ for $j = 1, 2, \ldots, n$. In addition to prices, he also looked at the labels and collected information about how much of the critical nutrients are contained in one serving of each food. Let us denote by $a_{ij}$ the amount of nutrient $i$ contained in food $j$. (Fortunately, he was able to call his favorite pizza delivery service and get similar information from them.) In terms of this information, he formulated the following linear programming problem:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \geq b_i & i = 1, 2, \ldots, m \\
& x_j \geq 0 & j = 1, 2, \ldots, n.
\end{array}
$$

Formulate the dual to this linear program. Can you introduce another person into the above story whose problem would naturally be to solve the dual?

**5.17** *Saddle points*. A function $h(y)$ defined for $y \in \mathbb{R}$ is called *strongly convex* if

- $h''(y) > 0$ for all $y \in \mathbb{R}$,
- $\lim_{y \to -\infty} h'(y) = -\infty$, and
- $\lim_{y \to \infty} h'(y) = \infty$.

A function $h$ is called *strongly concave* if $-h$ is strongly convex. Let $\pi(x, y)$, be a function defined for $(x, y) \in \mathbb{R}^2$ and having the following form

$$\pi(x, y) = f(x) - xy + g(y),$$

where $f$ is strongly concave and $g$ is strongly convex. Using elementary calculus

1. Show that there is one and only one point $(x^*, y^*) \in \mathbb{R}^2$ at which the gradient of $\pi$,
$$\nabla \pi = \begin{bmatrix} \partial\pi/\partial x \\ \partial\pi/\partial y \end{bmatrix},$$
vanishes. *Hint: From the two equations obtained by setting the derivatives to zero, derive two other relations having the form $x = \phi(x)$ and $y = \psi(y)$. Then study the functions $\phi$ and $\psi$ to show that there is one and only one solution.*
2. Show that
$$\max_{x \in \mathbb{R}} \min_{y \in \mathbb{R}} \pi(x, y) = \pi(x^*, y^*) = \min_{y \in \mathbb{R}} \max_{x \in \mathbb{R}} \pi(x, y),$$
where $(x^*, y^*)$ denotes the "critical point" identified in part 1 above. (Note: Be sure to check the signs of the second derivatives for both the inner and the outer optimizations.)

Associated with each strongly convex function $h$ is another function, called the *Legendre transform* of $h$ and denoted by $L_h$, defined by

$$L_h(x) = \max_{y \in \mathbb{R}} (xy - h(y)), \qquad x \in \mathbb{R}.$$

3. Using elementary calculus, show that $L_h$ is strongly convex.

4. Show that

$$\max_{x\in\mathbb{R}} \min_{y\in\mathbb{R}} \pi(x,y) = \max_{x\in\mathbb{R}}(f(x) - L_g(x))$$

and that

$$\min_{y\in\mathbb{R}} \max_{x\in\mathbb{R}} \pi(x,y) = \min_{y\in\mathbb{R}}(g(y) + L_{-f}(-y)).$$

5. Show that the Legendre transform of the Legendre transform of a function is the function itself. That is,

$$L_{L_h}(z) = h(z) \qquad \text{for all } z \in \mathbb{R}.$$

*Hint: This can be proved from scratch but it is easier to use the result of part 2 above.*

## Notes

The idea behind the strong duality theorem can be traced back to conversations between G.B. Dantzig and J. von Neumann in the fall of 1947, but an explicit statement did not surface until the paper of Gale et al. (1951). The term *primal problem* was coined by G.B. Dantzig's father, T. Dantzig. The dual simplex method was first proposed by Lemke (1954).

The solution to Exercise 5.13 (which is left to the reader to supply) suggests that a random linear programming problem is infeasible with probability $1/4$, unbounded with probability $1/4$, and has an optimal solution with probability $1/2$.