happens when we multiply $\bar{A}$ by $\hat{A}^T$ in both the permuted notation and the unpermuted notation:

$$\bar{A}\hat{A}^T = \begin{bmatrix} \bar{N} & \bar{B} \end{bmatrix} \begin{bmatrix} \hat{B}^T \\ \hat{N}^T \end{bmatrix} = \bar{N}\hat{B}^T + \bar{B}\hat{N}^T$$

and

$$\bar{A}\hat{A}^T = \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} -I \\ A \end{bmatrix} = -A + A = 0.$$

These two expressions obviously must agree so we see that

$$\bar{N}\hat{B}^T + \bar{B}\hat{N}^T = 0.$$

Putting the two terms on the opposite sides of the equality sign and multiplying on the right by the inverse of $\hat{B}^T$ and on the left by the inverse of $\bar{B}$, we get that

$$\bar{B}^{-1}\bar{N} = -\left(\hat{B}^{-1}\hat{N}\right)^T,$$

which is the property we wished to establish.

Exercises

6.1 Consider the following linear programming problem:

$$\begin{array}{lrcr} \text{maximize} & -6x_1 + 32x_2 - 9x_3 & & \\ \text{subject to} & -2x_1 + 10x_2 - 3x_3 & \leq & -6 \\ & x_1 - 7x_2 + 2x_3 & \leq & 4 \\ & x_1,\ x_2,\ x_3 & \geq & 0. \end{array}$$

Suppose that, in solving this problem, you have arrived at the following dictionary:

$$\begin{array}{rl} \zeta = & -18 - 3x_4 + 2x_2 \\ \hline x_3 = & 2 - x_4 + 4x_2 - 2x_5 \\ x_1 = & \phantom{-18 -{}} 2x_4 - x_2 + 3x_5. \end{array}$$

(a) Which variables are basic? Which are nonbasic?
(b) Write down the vector, $x^*_{\mathcal{B}}$, of current primal basic solution values.
(c) Write down the vector, $z^*_{\mathcal{N}}$, of current dual nonbasic solution values.
(d) Write down $B^{-1}N$.
(e) Is the primal solution associated with this dictionary feasible?
(f) Is it optimal?
(g) Is it degenerate?

6.2 Consider the following linear programming problem:

$$\begin{array}{lrcl} \text{maximize} & x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 & & \\ \text{subject to} & x_1 + 2x_2 + 3x_3 + 4x_4 + \ \ 5x_5 & \leq & 2 \\ & 7x_1 + 5x_2 - 3x_3 - 2x_4 \qquad\quad & \leq & 0 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 & \geq & 0. \end{array}$$

Consider the situation in which x_3 and x_5 are basic and all other variables are nonbasic. Write down:

(a) B,
(b) N,
(c) b,
(d) $c_{\mathcal{B}}$,
(e) $c_{\mathcal{N}}$,
(f) $B^{-1}N$,
(g) $x^*_{\mathcal{B}} = B^{-1}b$,
(h) $\zeta^* = c^T_{\mathcal{B}}B^{-1}b$,
(i) $z^*_{\mathcal{N}} = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}$,
(j) the dictionary corresponding to this basis.

6.3 Solve the problem in Exercise 2.1 using the matrix form of the primal simplex method.

6.4 Solve the problem in Exercise 2.4 using the matrix form of the dual simplex method.

6.5 Solve the problem in Exercise 2.3 using the two-phase approach in matrix form.

6.6 Find the dual of the following linear program:

$$\begin{array}{lrcccl} \text{maximize} & & & c^Tx & & \\ \text{subject to} & a & \leq & Ax & \leq & b \\ & l & \leq & x & \leq & u. \end{array}$$

6.7 (a) Let A be a given $m \times n$ matrix, c a given n-vector, and b a given m-vector. Consider the following max-min problem:

$$\max_{x \geq 0} \min_{y \geq 0} \left(c^T x - y^T A x + b^T y \right).$$

By noting that the inner optimization can be carried out explicitly, show that this problem can be reduced to a linear programming problem. Write it explicitly.

(b) What linear programming problem do you get if the min and max are interchanged?