ORF 307

Homework 8

AMPL Problems

Problem 2.

On Wednesday of finals week, there are 65 final exams all scheduled to take place during the 9am–noon time slot:

`http://orfe.princeton.edu/~rvdb/307/homework/courses.txt`

During this 3-hour time window, there are a total of 105 classrooms that could accomodate these exams:

`http://orfe.princeton.edu/~rvdb/307/homework/classrooms.txt`

The instructor for each class has been asked to pick (at least) five rooms from the list of available classrooms in which the final exam could be held. The instructor is also asked to give a preferencing rating for each of the five selected room choices. A preference of 100 means that the room is highly *desirable* while a preference of 0 is considered highly undesirable. Here's the data:

`http://orfe.princeton.edu/~rvdb/307/homework/course_class_desirability.txt`

The problem is to determine which room will hold the exam for each class so as to maximize the sum of all preferences. In other words, if we let $x_{i,j}$ be either zero or one where a 1 indicates that course i's exam will be held in classroom j, then the objective is to

$$\text{maximize} \sum_{(i,j)\in\text{Possibilities}} p_{i,j} x_{i,j},$$

where $p_{i,j}$ is the preference that the instructor for course i has given for having the final exam in room j. And, the constraints are that every course i must be assigned to exactly one classroom:

$$\sum_{j \text{ such that } (i,j) \text{ is a Possibility}} x_{i,j} = 1$$

and every classroom j can accomodate at most one course:

$$\sum_{i \text{ such that } (i,j) \text{ is a Possibility}} x_{i,j} \leq 1.$$

We also, of course, assume that every "flow" variable $x_{i,j}$ is non-negative and no larger than one.
Write an AMPL model to solve this problem using the data given in the links above.
Hint: It is strongly recommended that you read the chapter on modeling network flow problems in AMPL:

`http://ampl.com/BOOK/CHAPTERS/18-network.pdf`

Problem 3.

Part 1. The shortest distance between two points is a straight line. You may have learned this fact in school but I'm sure you all discovered it yourself probably before you were two years old. And, of course, Archimedes gets credit for being the first to articulate this axiom in writing. Or, was it Euclid? Hard to say.

Anyway, it's an optimization problem. This is an optimization class. Let's write an AMPL model to check it out.

Suppose at time $t = 0$ we are at an initial point (a_0, b_0, c_0). And, suppose that at time $t = 1$ we wish to be at a destination point (a_1, b_1, c_1). Let's let $(x(t), y(t), z(t))$ denote the path that we will follow from our initial point to our destination. As I'm sure you learned in one of your calculus courses, the length of this path is given by

$$\text{length} \quad = \quad \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \; dt.$$

We want to find the path that *minimizes* this length. In the world of optimization, this is called an *infinite dimensional* optimization problem because our decision variables are the x's, y's, and z's at each and every instant of time between $t = 0$ and $t = 1$. Let's simplify things by considering a finite approximation to this infinitely large problem. Specifically, lets break the time interval into n little intervals where n is some integer. Let's say $n = 50$. If our intervals are all of equal duration, then the duration of a single little time interval is $dt = 1/n$. And, the x coordinates of our path can now be thought of as $n+1$ values: $x_0, x_1, \ldots, x_n$. We have a similar finite collection of variables for the y and z part of the path's description. With these notations, we can formulate an approximation to the path length:

$$\text{approx. length} \quad = \quad \sum_{j=1}^{n} \sqrt{\left(\frac{x_j - x_{j-1}}{1/n}\right)^2 + \left(\frac{y_j - y_{j-1}}{1/n}\right)^2 + \left(\frac{z_j - z_{j-1}}{1/n}\right)^2} \; 1/n.$$

It turns out that there are many solutions to this problem. They all follow the same path but some go fast then slow while others go slow then fast etc. The exact temporal description of the path is not uniquely defined. This non-uniqueness can cause difficulties under some circumstances. If you run into trouble, you might consider fixing the x_j's (or the y_j's or the z_j's to some appropriate collection of values that moves from the starting value to the end value). Alternatively, you might consider requiring that the "speed" along the path is constant:

$$\sqrt{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 + (z_j - z_{j-1})^2}$$
$$= \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2 + (z_{j+1} - z_j)^2}, \qquad j = 1, 2, \ldots, n-1.$$

Another issue that often arises is the fact that the square-root function $f(x) = \sqrt{x}$ is not differentiable at $x = 0$. Yet, we optimization model involves lots of square roots. Difficulties with derivatives at zero can be addressed by adding a small positive constant, say 10^{-6}, inside the square roots. Such a small change could help.

Formulate the problem of minimizing this approximate length as an AMPL model. Pick a starting point and a destination that seem interesting to you. Solve the problem. (Technical note: you'll need to add this line

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option minos_options "superbasics_limit=500";
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just before the call to the solve command.) Check to see if the answer is a straight line.

Part 2. Here's a complication. We live on the Earth. The straight line path between New York and Paris would involve a tunnel that goes below the Atlantic ocean. That might be the shortest path, but it's not the most practical. So, let's add a constraint that the path must start, end, and remain throughout on the surface of the Earth:

$$x(t)^2 + y(t)^2 + z(t)^2 = r^2, \qquad 0 \le t \le 1.$$

Here r denotes Earth's radius ($r = 3959$ miles). Pick a starting point and a destination that lie on the surface of the Earth, add the above constraints to your finite dimensional approximation. Solve. Check to see if the shortest path is a great circle.