John Nash = A Beautiful Mind
Rock-Paper-Scissors

A two person game.

*Rules.* At the count of three declare one of:

Rock, Paper, Scissors

*Winner Selection.* Identical selection is a draw. Otherwise:

- Rock dulls Scissors
- Paper covers Rock
- Scissors cuts Paper

Check out Sam Kass’ version: Rock, Paper, Scissors, Lizard, Spock

It was featured on The Big Bang Theory.
Payoff Matrix

Payoffs are *from* row player *to* column player:

\[ A = \begin{bmatrix}
R & P & S \\
R & 0 & 1 & -1 \\
P & -1 & 0 & 1 \\
S & 1 & -1 & 0 \\
\end{bmatrix} \]

*Note:* Any *deterministic* strategy employed by either player can be defeated systematically by the other player.
Two-Person Zero-Sum Games

Given: $m \times n$ matrix $A$.

- **Row player** selects a strategy $i \in \{1, \ldots, m\}$.
- **Column player** selects a strategy $j \in \{1, \ldots, n\}$.
- Row player pays column player $a_{ij}$ dollars.

*Note:* The rows of $A$ represent deterministic strategies for row player, while columns of $A$ represent deterministic strategies for column player.

Deterministic strategies can be (and usually are) bad.
Randomized Strategies.

• Suppose row player picks $i$ with probability $y_i$.
• Suppose column player picks $j$ with probability $x_j$.

Throughout, $x = [x_1 \ x_2 \ \cdots \ x_n]^T$ and $y = [y_1 \ y_2 \ \cdots \ y_m]^T$ will denote stochastic vectors:

$$x_j \geq 0, \quad j = 1, 2, \ldots, n$$
$$\sum_j x_j = 1$$

$$y_i \geq 0, \quad i = 1, 2, \ldots, m$$
$$\sum_i y_i = 1$$

If row player uses random strategy $y$ and column player uses $x$, then expected payoff from row player to column player is

$$\sum_i \sum_j y_i a_{ij} x_j = y^T A x$$
Suppose column player were to adopt strategy $x$. Then, row player’s best defense is to use strategy $y$ that minimizes $y^T A x$: 

$$\min_y y^T A x$$

And so column player should choose that $x$ which maximizes these possibilities: 

$$\max_x \min_y y^T A x$$
What’s the solution to this problem:

\[
\begin{align*}
\text{minimize} & \quad 3y_1 + 6y_2 + 2y_3 + 18y_4 + 7y_5 \\
\text{subject to:} & \quad y_1 + y_2 + y_3 + y_4 + y_5 = 1 \\
& \quad y_i \geq 0, \quad i = 1, 2, 3, 4, 5
\end{align*}
\]
Inner optimization is easy:

\[ \min_y y^T Ax = \min_i e_i^T Ax \]

\((e_i\) denotes the vector that’s all zeros except for a one in the \(i\)-th position—that is, deterministic strategy \(i\)).

**Note:** Reduced a minimization over a *continuum* to one over a *finite set*.

We have:

\[ \max \left( \min_i e_i^T Ax \right) \]

\[ \sum_j x_j = 1, \quad x_j \geq 0, \quad j = 1, 2, \ldots, n. \]
Introduce a scalar variable $v$ representing the value of the inner minimization:

$$\max v$$

$$v \leq e_i^T Ax, \quad i = 1, 2, \ldots, m,$$

$$\sum_j x_j = 1,$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n.$$ 

Writing in pure matrix-vector notation:

$$\max v$$

$$ve - Ax \leq 0$$

$$e^T x = 1$$

$$x \geq 0$$

($e$ without a subscript denotes the vector of all ones).
Finally, in Block Matrix Form

\[
\begin{aligned}
\max & \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ v \end{bmatrix} \\
\begin{bmatrix} -A & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} & \leq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
x & \geq 0 \\
v & \text{free}
\end{aligned}
\]
Similarly, row player seeks $y^*$ attaining:

$$\min_y \max_x y^T Ax$$

which is equivalent to:

$$\min u$$
$$ue - A^T y \geq 0$$
$$e^T y = 1$$
$$y \geq 0$$
Row Player’s Problem in Block-Matrix Form

\[
\begin{align*}
\min & \quad \left[ \begin{array}{c}
0 \\
1
\end{array} \right]^T \left[ \begin{array}{c}
y \\
u
\end{array} \right] \\
\left[ \begin{array}{cc}
-A^T & e \\
e^T & 0
\end{array} \right] \left[ \begin{array}{c}
y \\
u
\end{array} \right] & \geq \left[ \begin{array}{c}
0 \\
1
\end{array} \right] \\
y & \geq 0 \\
u & \text{ free}
\end{align*}
\]

Note: Row player’s problem is dual to column player’s:

\[
\begin{align*}
\max & \quad \left[ \begin{array}{c}
0 \\
1
\end{array} \right]^T \left[ \begin{array}{c}
x \\
v
\end{array} \right] \\
\left[ \begin{array}{cc}
-A & e \\
e^T & 0
\end{array} \right] \left[ \begin{array}{c}
x \\
v
\end{array} \right] & \leq \left[ \begin{array}{c}
0 \\
1
\end{array} \right] \\
x & \geq 0 \\
v & \text{ free}
\end{align*}
\]

\[
\begin{align*}
\min & \quad \left[ \begin{array}{c}
0 \\
1
\end{array} \right]^T \left[ \begin{array}{c}
y \\
u
\end{array} \right] \\
\left[ \begin{array}{cc}
-A^T & e \\
e^T & 0
\end{array} \right] \left[ \begin{array}{c}
y \\
u
\end{array} \right] & \geq \left[ \begin{array}{c}
0 \\
1
\end{array} \right] \\
y & \geq 0 \\
u & \text{ free}
\end{align*}
\]
MiniMax Theorem

Theorem.

Let $x^*$ denote column player’s solution to her max–min problem. Let $y^*$ denote row player’s solution to his min–max problem. Then

$$\max_x y^T Ax = \min_y y^T Ax^*. $$

Proof. From \textit{Strong Duality Theorem}, we have

$$u^* = v^*. $$

Also,

$$v^* = \min_i e_i^T Ax^* = \min_y y^T Ax^*$$

$$u^* = \max_j y^* e_j = \max_x y^* T Ax$$

QED
set ROWS;
set COLS;
param A {ROWS,COLS} default 0;

var x{COLS} >= 0;
var v;

maximize zot: v;

subject to ineqs {i in ROWS}:
    sum{j in COLS} -A[i,j] * x[j] + v <= 0;

subject to equal:
    sum{j in COLS} x[j] = 1;
data;
set ROWS := P S R;
set COLS := P S R;
param A: P S R:=
    P 0 1 -2
    S -3 0 4
    R 5 -6 0
;
solve;
printf {j in COLS}: " %3s %10.7f 
", j, 102*x[j];
printf {i in ROWS}: " %3s %10.7f 
", i, 102*ineqs[i];
printf: "Value = %10.7f 
", 102*v;
ampl gamethy.mod
LOQO: optimal solution (12 iterations)
primal objective -0.1568627451
dual objective -0.1568627451
  P 40.0000000
  S 36.0000000
  R 26.0000000
  P 62.0000000
  S 27.0000000
  R 13.0000000
Value = -16.0000000
Consider:

\[
\begin{align*}
\text{max } c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\]

Rewrite equality constraints as pairs of inequalities:

\[
\begin{align*}
\text{max } c^T x \\
Ax &\leq b \\
-Ax &\leq -b \\
x &\geq 0
\end{align*}
\]

Put into block-matrix form:

\[
\begin{align*}
\text{max } c^T x \\
\begin{bmatrix}
A \\
-A
\end{bmatrix} x &\leq \\
\begin{bmatrix}
b \\
-b
\end{bmatrix}
\end{align*}
\]

Dual is:

\[
\begin{align*}
\text{min } & \begin{bmatrix}
b \\
-b
\end{bmatrix}^T \begin{bmatrix}
y^+ \\
y^-
\end{bmatrix} \\
\begin{bmatrix}
A^T & -A^T
\end{bmatrix} \begin{bmatrix}
y^+ \\
y^-
\end{bmatrix} &\geq c \\
y^+, y^- &\geq 0
\end{align*}
\]

Which is equivalent to:

\[
\begin{align*}
\text{min } & b^T (y^+ - y^-) \\
A^T (y^+ - y^-) &\geq c \\
y^+, y^- &\geq 0
\end{align*}
\]

Finally, letting \( y = y^+ - y^- \), we get

\[
\begin{align*}
\text{min } & b^T y \\
A^T y &\geq c \\
y &\text{ free.}
\end{align*}
\]
Summary

- Equality constraints $\implies$ free variables in dual.
- Inequality constraints $\implies$ nonnegative variables in dual.

Corollary:
- Free variables $\implies$ equality constraints in dual.
- Nonnegative variables $\implies$ inequality constraints in dual.
A Real-World Example

The Ultra-Conservative Investor

Consider again some historical investment data \((S_j(t))\):

As before, we can let \( R_{t,j} = S_j(t)/S_j(t-1) \) and view \( R \) as a payoff matrix in a game between Fate and the Investor.
The columns represent pure strategies for our conservative investor. The rows represent how history might repeat itself. Of course, for tomorrow, Fate won’t just repeat a previous day’s outcome but, rather, will present some mixture of these previous days. Likewise, the investor won’t put all of her money into one asset. Instead she will put a certain fraction into each. Using this data in the game-theory AMPL model, we get the following mixed-strategy percentages for Fate and for the investor.

**Investor’s Optimal Asset Mix:**

<table>
<thead>
<tr>
<th>Asset</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>XLP</td>
<td>98.4</td>
</tr>
<tr>
<td>XLU</td>
<td>1.6</td>
</tr>
</tbody>
</table>

**Mean Old Fate’s Mix:**

- 2011-08-08 55.9 ← Black Monday (2011)
- 2011-08-10 44.1

The value of the game is the investor’s expected return, 96.2%, which is actually a loss of 3.8%.

The data can be download from here: [http://finance.yahoo.com/q/hp?s=XLU](http://finance.yahoo.com/q/hp?s=XLU) Here, XLU is just one of the funds of interest.
To Ignore Black Monday (2011)
<table>
<thead>
<tr>
<th>Investor’s Optimal Asset Mix:</th>
<th>Mean Old Fate’s Mix:</th>
</tr>
</thead>
<tbody>
<tr>
<td>XLK  75.5</td>
<td>2015-03-25 3.9</td>
</tr>
<tr>
<td>XLV  15.9</td>
<td>2014-04-10 1.7</td>
</tr>
<tr>
<td>XLU  6.2</td>
<td>2013-06-20 68.9</td>
</tr>
<tr>
<td>XLB  2.2</td>
<td>2012-11-07 13.9</td>
</tr>
<tr>
<td>XLI  0.2</td>
<td>2012-06-01 11.5</td>
</tr>
</tbody>
</table>

The value of the game is the investor’s expected return, 97.7%, which is actually a loss of 2.3%.
Giving Fate Fewer Options

Thousands seemed unfair—How about 20...
Fate’s Conspiracy

Investor’s Optimal Asset Mix:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>MDY</td>
<td>83.7</td>
</tr>
<tr>
<td>XLE</td>
<td>13.2</td>
</tr>
<tr>
<td>XLF</td>
<td>3.2</td>
</tr>
</tbody>
</table>

Mean Old Fate’s Mix:

<table>
<thead>
<tr>
<th>Date</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>2015-03-25</td>
<td>11.5</td>
</tr>
<tr>
<td>2015-03-10</td>
<td>33.5</td>
</tr>
<tr>
<td>2015-03-06</td>
<td>55.0</td>
</tr>
</tbody>
</table>

The value of the game is the investor’s expected return, 98.7%, which is actually a loss of 1.3%.