ORF 307: Lecture 14

Linear Programming: Chapter 14: Network Flows: Algorithms

Robert J. Vanderbei

Apr 9, 2019

Slides last edited on January 25, 2019
Primal Network Simplex Method

Used when all primal flows are nonnegative (i.e., primal feasible).

Pivot Rules:

*Entering arc:* Pick a nontree arc having a negative (i.e. infeasible) dual slack.

*Leaving arc:* Add entering arc to make a cycle. Leaving arc is an arc on the cycle, pointing in the opposite direction to the entering arc, and of all such arcs, it is the one with the smallest primal flow.

Entering arc: (e,d)  
Leaving arc: (b,a)
Primal Method—Second Pivot

Entering arc: (c,b)
Leaving arc: (e,b)

Explanation of leaving arc rule:
- Increase flow on (c,b).
- Each unit increase produces a unit increase on arcs pointing in the same direction.
- Each unit increase produces a unit decrease on arcs pointing in the opposite direction.
- The first to reach zero will be the one pointing in the opposite direction and having the smallest flow among all such arcs.
Primal Method—Third Pivot

Entering arc: (a,d)
Leaving arc: (a,f)
Primal Method—Fourth Pivot

Entering arc: (c,a)
Leaving arc: (e,d)

Optimal!
Dual Network Simplex Method

Used when all dual slacks are nonnegative (i.e., dual feasible).

Pivot Rules:

Leaving arc: Pick a tree arc having a negative (i.e. infeasible) primal flow.

Leaving arc: (d,c)  Entering arc: (a,d)

Entering arc: Remove leaving arc to split the spanning tree into two subtrees. Entering arc is an arc reconnecting the spanning tree with an arc in the opposite direction, and, of all such arcs, is the one with the smallest dual slack.
Dual Network Simplex Method—Second Pivot

Leaving arc: (a,g)
Entering arc: (g,f)

Optimal!
Explanation of Entering Arc Rule

Recall initial tree solution:

Leaving arc: (d,c)
Entering arc: (a,d)

• Remove leaving arc. Need to find a reconnecting arc.
• Since the leaving arc has a negative flow, there is a net supply at the subtree attached to the head node and a net demand at the subtree attached to the tail node.
• So, reconnecting with an arc that spans in the same direction does not improve anything.
• Hence, only consider arcs spanning the two subtrees in the opposite direction.

• Consider a potential arc reconnecting in the opposite direction, say (a,f).
  - Its dual slack will drop to zero.
  - All other reconnecting arcs pointing in the same direction will drop by the same amount.
  - To maintain nonnegativity of all the others, must pick the one that drops the least.
Example.

- Turn off display of dual slacks.
- Turn on display of artificial dual slacks.
Two-Phase Method–First Pivot

Use dual network simplex method.
Leaving arc: (b,a)  Entering arc: (a,d)

Primal Feasible!
Two-Phase Method–Phase II

- Turn off display of artificial dual slacks.
- Turn on display of dual slacks.
Two-Phase Method—Second Pivot

Entering arc: \((h, c)\)
Leaving arc: \((a, c)\)
Two-Phase Method—Third Pivot

Entering arc: (e,d)
Leaving arc: (d,a)

Optimal!
Click here (or on any displayed network) to try out the online network simplex pivot tool.
• Artificial flows and slacks are multiplied by a parameter $\mu$.

• In the Figure, $4,1$ represents $4 + 1\mu$.

• **Question:** For which $\mu$ values is dictionary optimal?

• **Answer:**

\[
\begin{align*}
7 + \mu &\geq 0 \quad (a,c) \\
13 + \mu &\geq 0 \quad (a,d) \\
16 + \mu &\geq 0 \quad (a,h) \\
-8 + \mu &\geq 0 \quad (b,a) \\
-10 + \mu &\geq 0 \quad (b,c) \\
8 + \mu &\geq 0 \quad (b,g) \\
-8 + \mu &\geq 0 \quad (d,b) \\
\end{align*}
\]

\[
\begin{align*}
\mu &\geq 0 \quad (d,f) \\
15 + \mu &\geq 0 \quad (e,a) \\
4 + \mu &\geq 0 \quad (e,d) \\
5 + \mu &\geq 0 \quad (e,h) \\
12 + \mu &\geq 0 \quad (f,g) \\
2 + \mu &\geq 0 \quad (g,d) \\
-13 + \mu &\geq 0 \quad (h,c) \\
\end{align*}
\]

• That is, $13 \leq \mu < \infty$.

• Lower bound on $\mu$ is generated by arc $(h,c)$.

• Therefore, $(h,c)$ enters.

• Arc $(a,c)$ leaves.
Second Iteration

- Range of $\mu$ values: $8 \leq \mu \leq 13$.
- Leaving arc: (d,b)
- Entering arc: (e,d)

New tree is OPTIMAL!
Click here (or on any displayed network) to try out the online network simplex pivot tool.
Definition. *Network is called planar if can be drawn on a plane without intersecting arcs.*

Theorem. *Every planar network has a geometric dual—dual nodes are faces of primal network.*

Note:

- Primal spanning tree shown in red.

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*
Definition. *Network is called planar if can be drawn on a plane without intersecting arcs.*

Theorem. *Every planar network has a geometric dual—dual nodes are faces of primal network.*

Notes:
- Dual node $A$ is “node at infinity”.
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don’t forget node $A$).

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*
**Planar Networks (We Skipped This Part)**

**Definition.** *Network is called planar if can be drawn on a plane without intersecting arcs.*

**Theorem.** *Every planar network has a geometric dual—dual nodes are faces of primal network.*

**Notes:**
- Dual node $A$ is “node at infinity”.
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don’t forget node $A$).

**Theorem.** *A dual pivot on the primal network is exactly a primal pivot on the dual network.*
Theorem. Assuming integer supplies, every basic feasible solution assigns integer flow to every arc.

Corollary. Assuming integer supplies, every basic optimal solution assigns integer flow to every arc.