ORFEntertainment
Thursday
February 28th
7:30 PM
Sherrerd Hall Atrium

All are welcome, prizes, food, fun

Performance times are still available... it's not too late to sign up!!!

New this year karaoke!!!

Contact tzigler@princeton.edu
Complementary Slackness

Primal Problem:

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j + w_i = b_i & i = 1, \ldots, m \\
& \quad x_j \geq 0 & j = 1, \ldots, n \\
& \quad w_i \geq 0 & i = 1, \ldots, m
\end{align*}
\]

Dual Problem:

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} y_i a_{ij} - z_j = c_j & j = 1, \ldots, n \\
& \quad y_i \geq 0 & i = 1, \ldots, m \\
& \quad z_j \geq 0 & j = 1, \ldots, n
\end{align*}
\]

Theorem. At optimality, we have

\[
\begin{align*}
x_j z_j &= 0, \quad \text{for } j = 1, 2, \ldots, n, \\
w_i y_i &= 0, \quad \text{for } i = 1, 2, \ldots, m.
\end{align*}
\]
Recall the proof of the Weak Duality Theorem:

\[
\sum_j c_j x_j \leq \sum_j (c_j + z_j) x_j = \sum_j \left( \sum_i y_i a_{ij} \right) x_j = \sum_{ij} y_i a_{ij} x_j
\]

\[
= \sum_i \left( \sum_j a_{ij} x_j \right) y_i = \sum_i (b_i - w_i) y_i \leq \sum_i b_i y_i,
\]

The inequalities come from the fact that

\[
\begin{align*}
x_j z_j & \geq 0, \quad \text{for all } j, \\
w_i y_i & \geq 0, \quad \text{for all } i.
\end{align*}
\]

By Strong Duality Theorem, the inequalities are equalities at optimality.
Dual Simplex Method
Dual Simplex Method

When: dual feasible, primal infeasible (i.e., pinks on the left, not on top).

An Example. Showing both primal and dual dictionaries:

Looking at dual dictionary: $y_3$ enters, $z_2$ leaves.

On the primal dictionary: $w_3$ leaves, $x_2$ enters.

After pivot...

(Seed = 3, generate 3x, with negation)
Going in, we have:

Looking at dual: \( y_2 \) enters, \( z_1 \) leaves.

Looking at primal: \( w_2 \) leaves, \( x_1 \) enters.
Referring to the primal dictionary:

- Pick leaving variable from those rows that are *infeasible*.
- Pick entering variable from a box with a negative value and which can be increased the least (on the dual side).

Next primal dictionary shown on next page...
Going in, we have:

<table>
<thead>
<tr>
<th>maximize $\zeta$</th>
<th>-22</th>
<th>+</th>
<th>-2</th>
<th>$w_2$</th>
<th>+</th>
<th>-9/2</th>
<th>$w_3$</th>
<th>+</th>
<th>-27</th>
<th>$x_3$</th>
<th>+</th>
<th>-21/2</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>-6</td>
<td>-</td>
<td>1/2</td>
<td>$w_2$</td>
<td>-</td>
<td>5/4</td>
<td>$w_3$</td>
<td>-</td>
<td>8</td>
<td>$x_3$</td>
<td>-</td>
<td>-7/4</td>
<td>$x_4$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>2</td>
<td>-</td>
<td>-1/4</td>
<td>$w_2$</td>
<td>-</td>
<td>-3/8</td>
<td>$w_3$</td>
<td>-</td>
<td>-2</td>
<td>$x_3$</td>
<td>-</td>
<td>-3/8</td>
<td>$x_4$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>-</td>
<td>1/4</td>
<td>$w_2$</td>
<td>-</td>
<td>-1/8</td>
<td>$w_3$</td>
<td>-</td>
<td>-1</td>
<td>$x_3$</td>
<td>-</td>
<td>-1/8</td>
<td>$x_4$</td>
</tr>
</tbody>
</table>

Which variable must leave and which must enter?

See next page...
Answer is: $w_1$ leaves, $x_4$ enters.

Resulting dictionary is OPTIMAL:
Dual-Based Phase I Method
Example:

<table>
<thead>
<tr>
<th>maximize ( \zeta )</th>
<th>( 0 )</th>
<th>+</th>
<th>( 3 )</th>
<th>( x_1 )</th>
<th>+</th>
<th>( 6 )</th>
<th>( x_2 )</th>
<th>+</th>
<th>( -6 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta_0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( w_1 )</td>
<td>-2</td>
<td>-</td>
<td>1</td>
<td>( x_1 )</td>
<td>-</td>
<td>-1</td>
<td>( x_2 )</td>
<td>-</td>
<td>-2</td>
<td>( x_3 )</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>-1</td>
<td>-</td>
<td>0</td>
<td>( x_1 )</td>
<td>-</td>
<td>1</td>
<td>( x_2 )</td>
<td>-</td>
<td>-2</td>
<td>( x_3 )</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>6</td>
<td>-</td>
<td>1</td>
<td>( x_1 )</td>
<td>-</td>
<td>4</td>
<td>( x_2 )</td>
<td>-</td>
<td>0</td>
<td>( x_3 )</td>
</tr>
<tr>
<td>( w_4 )</td>
<td>3</td>
<td>-</td>
<td>2</td>
<td>( x_1 )</td>
<td>-</td>
<td>1</td>
<td>( x_2 )</td>
<td>-</td>
<td>5</td>
<td>( x_3 )</td>
</tr>
</tbody>
</table>

\( x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0 \)

Seed = 4

Notes:

- Two objective functions: the true objective (on top), and a fake one (below it).
- For Phase I, use the fake objective—it’s dual feasible.

Phase I—First Pivot: \( w_1 \) leaves, \( x_3 \) enters.

Let’s go pivoting...
Recall initial dictionary:

Dual pivot: $w_1$ leaves, $x_3$ enters.

After pivot:
Recall current dictionary:

\[
\begin{align*}
\text{maximize } \zeta &= -6 + 0 x_1 + 9 x_2 + -3 w_1 \\
\zeta_0 &= -3/2 x_1 + -1/2 x_2 + -1/2 w_1 \\
x_3 &= 1 - -1/2 x_1 - 1/2 x_2 - -1/2 w_1 \\
w_2 &= 1 - -1 x_1 - 2 x_2 - -1 w_1 \\
w_3 &= 6 - 1 x_1 - 4 x_2 - 0 w_1 \\
w_4 &= -2 - 1/2 x_1 - -7/2 x_2 - 5/2 w_1
\end{align*}
\]

\[x_1, x_2, x_3, w_1, w_2, w_3, w_4 \geq 0\]

Dual pivot: \(w_4\) leaves, \(x_2\) enters.

After pivot:
Recall current dictionary:

Dual pivot: $w_2$ leaves, $x_1$ enters.

After pivot:

Feasible!
Current dictionary is feasible:

```
maximize ζ = -3/5 + 9/5 w_2 + 18/5 w_4 + 21/5 w_1
ζ_0 = -11/5 w_2 + -7/5 w_4 + -9/5 w_1

x_3 = 4/5 - -3/5 w_2 - -1/5 w_4 - -2/5 w_1
x_1 = 1/5 - -7/5 w_2 - -4/5 w_4 - -3/5 w_1
w_3 = 17/5 - 11/5 w_2 - 12/5 w_4 - 19/5 w_1
x_2 = 3/5 - -1/5 w_2 - -2/5 w_4 - -4/5 w_1

x_1, x_2, x_3, w_1, w_2, w_3, w_4 ≥ 0
```

Ignore fake objective. Use the real objective. Primal pivot: $w_1$ enters, $w_3$ leaves.

After pivot:

```
maximize ζ = 60/19 + -12/19 w_2 + 18/19 w_4 + -21/19 w_3
ζ_0 = -22/19 w_2 + -5/19 w_4 + 9/19 w_3

x_3 = 22/19 - -7/19 w_2 - 1/19 w_4 - 2/19 w_3
x_1 = 14/19 - -20/19 w_2 - -8/19 w_4 - 3/19 w_3
w_1 = 17/19 - 11/19 w_2 - 12/19 w_4 - 5/19 w_3
x_2 = 25/19 - 5/19 w_2 - 2/19 w_4 - 4/19 w_3

x_1, x_2, x_3, w_1, w_2, w_3, w_4 ≥ 0
```
Getting close:

Primal pivot: $w_4$ enters, $w_1$ leaves.

After pivot:

Optimal!