# Linear Programming: Chapter 5
# Duality

Robert J. Vanderbei

September 30, 2010

Slides last edited on October 5, 2010

Operations Research and Financial Engineering
Princeton University
Princeton, NJ 08544
http://www.princeton.edu/~rvdb

# Resource Allocation

Recall the resource allocation problem ($m = 2$, $n = 3$):

$$\begin{array}{lrcrcrcl} \text{maximize} & c_1x_1 & + & c_2x_2 & + & c_3x_3 & & \\ \text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & \leq & b_1 \\ & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & \leq & b_2 \\ & & & & & x_1,\ x_2,\ x_3 & \geq & 0, \end{array}$$

where

$$\begin{aligned} c_j &= \text{profit per unit of product } j \text{ produced} \\ b_i &= \text{units of raw material } i \text{ on hand} \\ a_{ij} &= \text{units raw material } i \text{ required to produce } 1 \text{ unit of prod } j. \end{aligned}$$

# Closing Up Shop

If we produce one unit less of product $j$, then we free up:

- $a_{1j}$ units of raw material $1$ and
- $a_{2j}$ units of raw material $2$.

Selling these unused raw materials for $y_1$ and $y_2$ dollars/unit yields $a_{1j}y_1 + a_{2j}y_2$ dollars.

Only interested if this exceeds lost profit on each product $j$:

$$a_{1j}y_1 + a_{2j}y_2 \geq c_j, \qquad j = 1, 2, 3.$$

Consider a buyer offering to purchase our entire inventory.

Subject to above constraints, buyer wants to minimize cost:

$$\begin{array}{lrcrcl}
\text{minimize} & b_1 y_1 & + & b_2 y_2 & & \\
\text{subject to} & a_{11} y_1 & + & a_{21} y_2 & \geq & c_1 \\
& a_{12} y_1 & + & a_{22} y_2 & \geq & c_2 \\
& a_{13} y_1 & + & a_{23} y_2 & \geq & c_3 \\
& & & y_1, \; y_2 & \geq & 0 \; .
\end{array}$$

# Duality

Every Problem:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, 2, \ldots, m \\ & x_j \geq 0 & j = 1, 2, \ldots, n, \end{array}$$

Has a Dual:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} y_i a_{ij} \geq c_j & j = 1, 2, \ldots, n \\ & y_i \geq 0 & i = 1, 2, \ldots, m. \end{array}$$

# Dual of Dual

## Primal Problem:

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & i = 1, \ldots, m \\ & x_j \ge 0 & j = 1, \ldots, n, \end{array}$$

Original problem is called the *primal problem*.

A problem is defined by its data (notation used for the variables is arbitrary).

## Dual in "Standard" Form:

$$\begin{array}{lll} -\text{maximize} & \sum_{i=1}^{m} -b_i y_i & \\ \text{subject to} & \sum_{i=1}^{m} -a_{ij} y_i \le -c_j & j = 1, \ldots, n \\ & y_i \ge 0 & i = 1, \ldots, m. \end{array}$$

Dual is "negative transpose" of primal.

**Theorem** *Dual of dual is primal.*

# Weak Duality Theorem

*If* $(x_1, x_2, \ldots, x_n)$ *is feasible for the primal and* $(y_1, y_2, \ldots, y_m)$ *is feasible for the dual, then*

$$\sum_j c_j x_j \le \sum_i b_i y_i.$$

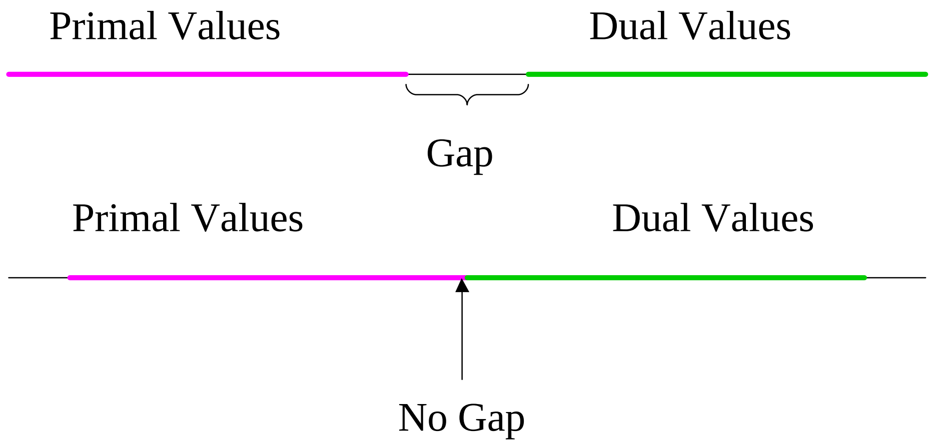
**Proof.**

$$\begin{aligned} \sum_j c_j x_j &\le \sum_j \left( \sum_i y_i a_{ij} \right) x_j \\ &= \sum_{ij} y_i a_{ij} x_j \\ &= \sum_i \left( \sum_j a_{ij} x_j \right) y_i \\ &\le \sum_i b_i y_i. \end{aligned}$$

# Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?




Answer is provided by the Strong Duality Theorem (coming later).

# Simplex Method and Duality

*A Primal Problem:*

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 0 | + | -3 | x1 | + | 2 | x2 | + | 1 | x3 |
| w1 | = | 0 | - | 0 | x1 | - | -1 | x2 | - | 2 | x3 |
| w2 | = | 3 | - | -3 | x1 | - | 4 | x2 | - | 1 | x3 |

*Its Dual:*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| obj | = | 0 | + | 0 | y1 | + | -3 | y2 |
| z1 | = | 3 | - | 0 | y1 | - | 3 | y2 |
| z2 | = | -2 | - | 1 | y1 | - | -4 | y2 |
| z3 | = | -1 | - | -2 | y1 | - | -1 | y2 |

Notes:

- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: $x_2$ enters, $w_2$ leaves.
Make analogous pivot in dual: $z_2$ leaves, $y_2$ enters.

# Second Iteration

*After First Pivot:*

Primal (feasible):

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 3/2 | + | -3/2 | x1 | + | -1/2 | w2 | + | 1/2 | x3 |
| w1 | = | 3/4 | - | -3/4 | x1 | - | 1/4 | w2 | - | 9/4 | x3 |
| x2 | = | 3/4 | - | -3/4 | x1 | - | 1/4 | w2 | - | 1/4 | x3 |

Dual (still not feasible):

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| obj | = | -3/2 | + | -3/4 | y1 | + | -3/4 | z2 |
| z1 | = | 3/2 | - | 3/4 | y1 | - | 3/4 | z2 |
| y2 | = | 1/2 | - | -1/4 | y1 | - | -1/4 | z2 |
| z3 | = | -1/2 | - | -9/4 | y1 | - | -1/4 | z2 |

Note: *negative transpose property intact.*

Again, use primal to pick pivot: $x_3$ enters, $w_1$ leaves.

Make analogous pivot in dual: $z_3$ leaves, $y_1$ enters.

# After Second Iteration

Primal:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 5/3 | + | -4/3 | x1 | + | -5/9 | w2 | + | -2/9 | w1 |
| x3 | = | 1/3 | - | -1/3 | x1 | - | 1/9 | w2 | - | 4/9 | w1 |
| x2 | = | 2/3 | - | -2/3 | x1 | - | 2/9 | w2 | - | -1/9 | w1 |

- Is *optimal*.

Dual:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| obj | = | -5/3 | + | -1/3 | z3 | + | -2/3 | z2 |
| z1 | = | 4/3 | - | 1/3 | z3 | - | 2/3 | z2 |
| y2 | = | 5/9 | - | -1/9 | z3 | - | -2/9 | z2 |
| y1 | = | 2/9 | - | -4/9 | z3 | - | 1/9 | z2 |

- Negative transpose property remains intact.
- Is *optimal*.

## Conclusion

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.

# Strong Duality Theorem

Conclusion on previous slide is the essence of the *strong duality theorem* which we now state:

**Theorem.** *If the primal problem has an optimal solution,*

$$x^* = (x_1^*, x_2^*, \ldots, x_n^*),$$

*then the dual also has an optimal solution,*

$$y^* = (y_1^*, y_2^*, \ldots, y_m^*),$$

*and*

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

**Paraphrase:**

If primal has an optimal solution, then there is no duality gap.

# Duality Gap

## Four possibilities:

- Primal optimal, dual optimal (no gap).
- Primal unbounded, dual infeasible (no gap).
- Primal infeasible, dual unbounded (no gap).
- Primal infeasible, dual infeasible (infinite gap).

## Example of *infinite gap*:

$$
\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & x_1 & - & x_2 & \leq & 1 \\
& -x_1 & + & x_2 & \leq & -2 \\
& & x_1, & x_2 & \geq & 0.
\end{array}
$$

# Complementary Slackness

**Theorem.** *At optimality, we have*

$$\begin{aligned} x_j z_j &= 0, \qquad \textit{for } j = 1, 2, \ldots, n, \\ w_i y_i &= 0, \qquad \textit{for } i = 1, 2, \ldots, m. \end{aligned}$$

# Proof

Recall the proof of the Weak Duality Theorem:

$$
\begin{aligned}
\sum_j c_j x_j \;&\le\; \sum_j (c_j + z_j) x_j = \sum_j \left( \sum_i y_i a_{ij} \right) x_j = \sum_{ij} y_i a_{ij} x_j \\
&=\; \sum_i \left( \sum_j a_{ij} x_j \right) y_i = \sum_i (b_i - w_i) y_i \le \sum_i b_i y_i,
\end{aligned}
$$

The inequalities come from the fact that

$$
\begin{aligned}
x_j z_j \;&\ge\; 0, \quad \text{for all } j, \\
w_i y_i \;&\ge\; 0, \quad \text{for all } i.
\end{aligned}
$$

By Strong Duality Theorem, the inequalities are equalities at optimality.

# Dual Simplex Method

When: dual feasible, primal infeasible (i.e., pinks on the left, not on top).

*An Example.* Showing both primal and dual dictionaries:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 0.0 | + | -2.0 | x1 | + | -4.0 | x2 | + | 0.0 | x3 | + | -6.0 | x4 | |
| w1 | = | -3.0 | - | -1.0 | x1 | - | 2.0 | x2 | - | 0.0 | x3 | - | -1.0 | x4 | |
| w2 | = | -5.0 | - | 2.0 | x1 | - | -3.0 | x2 | - | 0.0 | x3 | - | -2.0 | x4 | |
| w3 | = | 8.0 | - | 2.0 | x1 | - | 3.0 | x2 | - | 3.0 | x3 | - | 2.0 | x4 | |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 0.0 | + | 3.0 | y1 | + | 5.0 | y2 | + | -8.0 | y3 |
| z1 | = | 2.0 | - | 1.0 | y1 | - | -2.0 | y2 | - | -2.0 | y3 |
| z2 | = | 4.0 | - | -2.0 | y1 | - | 3.0 | y2 | - | -3.0 | y3 |
| z3 | = | 0.0 | - | 0.0 | y1 | - | 0.0 | y2 | - | -3.0 | y3 |
| z4 | = | 6.0 | - | 1.0 | y1 | - | 2.0 | y2 | - | -2.0 | y3 |

Looking at dual dictionary: $y_2$ enters, $z_2$ leaves.

On the primal dictionary: $w_2$ leaves, $x_2$ enters.

After pivot...

# Dual Simplex Method: Second Pivot

Going in, we have:

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | -6.6667 | + | -4.6667 | x1 | + | -1.3333 | w2 | + | 0.0 | x3 | + | -3.3333 | x4 |
| w1 | = | -6.3333 | - | 0.3333 | x1 | - | 0.6667 | w2 | - | 0.0 | x3 | - | -2.3333 | x4 |
| x2 | = | 1.6667 | - | -0.6667 | x1 | - | -0.3333 | w2 | - | 0.0 | x3 | - | 0.6667 | x4 |
| w3 | = | 3.0 | - | 4.0 | x1 | - | 1.0 | w2 | - | 3.0 | x3 | - | 0.0 | x4 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 6.6667 | + | 6.3333 | y1 | + | -1.6667 | z2 | + | -3.0 | y3 |
| z1 | = | 4.6667 | - | -0.3333 | y1 | - | 0.6667 | z2 | - | -4.0 | y3 |
| y2 | = | 1.3333 | - | -0.6667 | y1 | - | 0.3333 | z2 | - | -1.0 | y3 |
| z3 | = | 0.0 | - | 0.0 | y1 | - | 0.0 | z2 | - | -3.0 | y3 |
| z4 | = | 3.3333 | - | 2.3333 | y1 | - | -0.6667 | z2 | - | 0.0 | y3 |

Looking at dual: $y_1$ enters, $z_4$ leaves.

Looking at primal: $w_1$ leaves, $x_4$ enters.

# Dual Simplex Method Pivot Rule

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | -6.6667 | + | -4.6667 | x1 | + | -1.3333 | w2 | + | 0.0 | x3 | + | -3.3333 | x4 |
| w1 | = | -6.3333 | - | 0.3333 | x1 | - | 0.6667 | w2 | - | 0.0 | x3 | - | -2.3333 | x4 |
| x2 | = | 1.6667 | - | -0.6667 | x1 | - | -0.3333 | w2 | - | 0.0 | x3 | - | 0.6667 | x4 |
| w3 | = | 3.0 | - | 4.0 | x1 | - | 1.0 | w2 | - | 3.0 | x3 | - | 0.0 | x4 |

Refering to the primal dictionary:

- Pick leaving variable from those rows that are *infeasible*.
- Pick entering variable from a box with a negative value and which can be increased the least (on the dual side).

Next primal dictionary shown on next page...

# Dual Simplex Method: Third Pivot

Going in, we have:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | -15.7143 | + | -5.1429 | x1 | + | -2.2857 | w2 | + | 0.0 | x3 | + | -1.4286 | w1 |
| x4 | = | 2.7143 | − | -0.1429 | x1 | − | -0.2857 | w2 | − | 0.0 | x3 | − | -0.4286 | w1 |
| x2 | = | -0.1429 | − | -0.5714 | x1 | − | -0.1429 | w2 | − | 0.0 | x3 | − | 0.2857 | w1 |
| w3 | = | 3.0 | − | 4.0 | x1 | − | 1.0 | w2 | − | 3.0 | x3 | − | 0.0 | w1 |

Which variable must leave and which must enter?

See next page...

# Dual Simplex Method: Third Pivot—Answer

Answer is: $x_2$ leaves, $x_1$ enters.

Resulting dictionary is OPTIMAL:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | -17.0 | + | -9.0 | x2 | + | -1.0 | w2 | + | 0.0 | x3 | + | -4.0 | w1 |
| x4 | = | 2.75 | - | -0.25 | x2 | - | -0.25 | w2 | - | 0.0 | x3 | - | -0.5 | w1 |
| x1 | = | 0.25 | - | -1.75 | x2 | - | 0.25 | w2 | - | 0.0 | x3 | - | -0.5 | w1 |
| w3 | = | 2.0 | - | 7.0 | x2 | - | 0.0 | w2 | - | 3.0 | x3 | - | 2.0 | w1 |

# Dual-Based Phase I Method

*Example:*

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 0.0 | | | + | -4.0 | x1 | + | 2.0 | x2 | + | 3.0 | x3 |
| | | | | | + | -1.0 | x1 | + | -1.0 | x2 | + | -1.0 | x3 |
| w1 | = | 0.0 | + | 1.0 | − | 2.0 | x1 | − | -1.0 | x2 | − | 3.0 | x3 |
| w2 | = | 0.0 | + | 1.0 | − | 3.0 | x1 | − | -3.0 | x2 | − | -4.0 | x3 |
| w3 | = | -3.0 | + | 1.0 | − | -1.0 | x1 | − | -1.0 | x2 | − | 1.0 | x3 |
| w4 | = | -1.0 | + | 1.0 | − | -2.0 | x1 | − | 0.0 | x2 | − | 0.0 | x3 |

Notes:

- Two objective functions: the true objective (on top), and a fake one (below it).
- For *Phase I*, use the fake objective—it's dual feasible.
- Two right-hand sides: the real one (on the left) and a fake (on the right).
- Ignore the fake right-hand side—we'll use it in another algorithm later.

*Phase I—First Pivot:* $w_3$ leaves, $x_1$ enters.
After first pivot...

# Dual-Based Phase I Method—Second Pivot

Recall current dictionary:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | -12.0 | | | + | -4.0 | w3 | + | 6.0 | x2 | + | -1.0 | x3 |
| | | | | | + | -1.0 | w3 | + | 0.0 | x2 | + | -2.0 | x3 |
| w1 | = | -6.0 | + | 3.0 | − | 2.0 | w3 | − | -3.0 | x2 | − | 5.0 | x3 |
| w2 | = | -9.0 | + | 4.0 | − | 3.0 | w3 | − | -6.0 | x2 | − | -1.0 | x3 |
| x1 | = | 3.0 | + | -1.0 | − | -1.0 | w3 | − | 1.0 | x2 | − | -1.0 | x3 |
| w4 | = | 5.0 | + | -1.0 | − | -2.0 | w3 | − | 2.0 | x2 | − | -2.0 | x3 |

Dual pivot: $w_2$ leaves, $x_2$ enters.

After pivot:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | -3.0 | | | + | -1.0 | w3 | + | 1.0 | w2 | + | -2.0 | x3 |
| | | | | | + | -1.0 | w3 | + | 0.0 | w2 | + | -2.0 | x3 |
| w1 | = | -1.5 | + | 1.0 | − | 0.5 | w3 | − | -0.5 | w2 | − | 5.5 | x3 |
| x2 | = | 1.5 | + | -0.6667 | − | -0.5 | w3 | − | -0.1667 | w2 | − | 0.1667 | x3 |
| x1 | = | 1.5 | + | -0.3333 | − | -0.5 | w3 | − | 0.1667 | w2 | − | -1.1667 | x3 |
| w4 | = | 2.0 | + | 0.3333 | − | -1.0 | w3 | − | 0.3333 | w2 | − | -2.3333 | x3 |

# Dual-Based Phase I Method—Third Pivot

Current dictionary:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | -3.0 | | | + | -1.0 | w3 | + | 1.0 | w2 | + | -2.0 | x3 |
| | | | | | + | -1.0 | w3 | + | 0.0 | w2 | + | -2.0 | x3 |
| w1 | = | -1.5 | + | 1.0 | − | 0.5 | w3 | − | -0.5 | w2 | − | 5.5 | x3 |
| x2 | = | 1.5 | + | -0.6667 | − | -0.5 | w3 | − | -0.1667 | w2 | − | 0.1667 | x3 |
| x1 | = | 1.5 | + | -0.3333 | − | -0.5 | w3 | − | 0.1667 | w2 | − | -1.1667 | x3 |
| w4 | = | 2.0 | + | 0.3333 | − | -1.0 | w3 | − | 0.3333 | w2 | − | -2.3333 | x3 |

Dual pivot:
- $w_1$ leaves,
- $w_2$ enters.

After pivot:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 0.0 | | | + | 0.0 | w3 | + | 2.0 | w1 | + | 9.0 | x3 |
| | | | | | + | -1.0 | w3 | + | 0.0 | w1 | + | -2.0 | x3 |
| w2 | = | 3.0 | + | -2.0 | − | -1.0 | w3 | − | -2.0 | w1 | − | -11.0 | x3 |
| x2 | = | 2.0 | + | -1.0 | − | -0.6667 | w3 | − | -0.3333 | w1 | − | -1.6667 | x3 |
| x1 | = | 1.0 | + | 0.0 | − | -0.3333 | w3 | − | 0.3333 | w1 | − | 0.6667 | x3 |
| w4 | = | 1.0 | + | 1.0 | − | -0.6667 | w3 | − | 0.6667 | w1 | − | 1.3333 | x3 |

It's **feasible!**

# Fourth Pivot—Phase II

Current dictionary:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 0.0 | | | + | 0.0 | w3 | + | 2.0 | w1 | + | 9.0 | x3 |
| | | | | | + | -1.0 | w3 | + | 0.0 | w1 | + | -2.0 | x3 |
| w2 | = | 3.0 | + | -2.0 | − | -1.0 | w3 | − | -2.0 | w1 | − | -11.0 | x3 |
| x2 | = | 2.0 | + | -1.0 | − | -0.6667 | w3 | − | -0.3333 | w1 | − | -1.6667 | x3 |
| x1 | = | 1.0 | + | 0.0 | − | -0.3333 | w3 | − | 0.3333 | w1 | − | 0.6667 | x3 |
| w4 | = | 1.0 | + | 1.0 | − | -0.6667 | w3 | − | 0.6667 | w1 | − | 1.3333 | x3 |

It's feasible.

Ignore fake objective.

Use the real thing (top row).

Primal pivot: $x_3$ enters, $w_4$ leaves.

# Final Dictionary

After pivot:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| obj | = | 6.75 | | | + | 4.5 | w3 | + | -2.5 | w1 | + | -6.75 | w4 |
| | | | | | + | -2.0 | w3 | + | 1.0 | w1 | + | 1.5 | w4 |
| w2 | = | 11.25 | + | 6.25 | − | -6.5 | w3 | − | 3.5 | w1 | − | 8.25 | w4 |
| x2 | = | 3.25 | + | 0.25 | − | -1.5 | w3 | − | 0.5 | w1 | − | 1.25 | w4 |
| x1 | = | 0.5 | + | -0.5 | − | 0.0 | w3 | − | 0.0 | w1 | − | -0.5 | w4 |
| x3 | = | 0.75 | + | 0.75 | − | -0.5 | w3 | − | 0.5 | w1 | − | 0.75 | w4 |

Problem is **unbounded!**