Princeton University
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Optimization

Qualifying Exam

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Instructions: You may use any written reference you wish for this exam. You must however acknowledge any reference used.

Brevity and clarity will be rewarded. Especially brevity.

In financial markets, an American option is a contract giving the holder of the option the right to buy a particular stock at a particular price S, called the *strike price*, at any time in the future (in the real-world, there is an expiration date; for simplicity we will assume that the right never expires). Let X_n denote the share price of the stock at a time n days into the future. Obviously, X_n is a random variable and the collection of these random variables over time is a stochastic process. Let us assume that X_n is a random walk on the state space $E = \{j\Delta x : j = 0, 1, 2, \ldots\}$, where Δx is a fixed small positive real number. That is, if the current stock price is $x \in E$, then the price tomorrow will be either $x + \Delta x$ or $x - \Delta x$ with probabilities p and $q = 1 - p$, respectively.

In order to figure out how to price the option, one must answer the question: what is the expected value, in today's dollars, of the option? Naturally, the seller of the option should assume that the buyer will employ an optimal strategy for exercising the option. If at some future date, the stock price X_n is larger than S, then the holder of the option can buy the stock for S dollars and immediately sell it for X_n dollars and realize a gain of $X_n - S$ dollars. If, on the other hand, the price is less than S, then there is no value in exercising the option—the option holder would only lose money needlessly. Hence, the value at time n of the option is

$$f(X_n) = \begin{cases} X_n - S & \text{if } X_n \geq S \\ 0 & \text{if } X_n \leq S. \end{cases}$$

If the option holder exercises the option at some time τ (possibly random, but not clairvoyant), then the expected present value would be

$$\mathbf{E}_x \alpha^\tau f(X_\tau).$$

Here we have introduce a *discount factor* α (a number slightly less than one). This discount factor accounts for the fact that future dollars are worth less than present dollars. Specifically, α gives the today's value of tomorrow's dollar. The optimal strategy is then determined by maximizing over all non-clairvoyant random times τ:

$$v(x) = \max_\tau \mathbf{E}_x \alpha^\tau f(X_\tau).$$

The function v is called the *value function*. It tells both the buyer and the seller of the option everything they need to know. It tells the seller that the "fair" price for the option is $v(x)$ if the

current stock price is x. It also characterizes the optimal strategy for the buyer of the option. The buyer should not exercise the option at times n when $v(X_n) > f(X_n)$. In fact, he/she should exercise the option exactly at the first time that $v(X_n) = f(X_n)$.

In a probability course, it is shown that v can be uniquely characterized as the smallest function that satisfies these inequalities:

$$v(x) \geq f(x), \qquad\qquad\qquad x \in E$$
$$v(x) \geq \alpha\left(pv(x+\Delta x) + qv(x-\Delta x)\right), \quad x \in E \setminus \{0\}.$$

Recalling the discreteness of our model's state space, we can formulate an infinite dimensional linear programming problem

$$\text{minimize} \quad \sum_{j=0}^{\infty} v_j$$
$$\text{subject to} \quad v_j \geq f_j \qquad\qquad\qquad j = 0, 1, 2, \dots$$
$$\qquad\qquad\quad v_j \geq \alpha\left(pv_{j+1} + qv_{j-1}\right) \quad j = 1, 2, \dots$$

where $x_j = j\Delta x$, $v_j = v(x_j)$, and $f_j = f(x_j)$.

Let v_j^*, $j = 0, 1, \dots$, denote the optimal solution to this linear programming problem. From the original real-world description of the problem, it would seem that the optimal strategy for exercizing the option would be not to exercize when the stock price is below some threshold and then exercise as soon as the stock price hits the threshold value. In other words, it would seem that...

>   *There exists a j^* such that*

$$v_0^* = f_0,$$
$$v_j^* = \alpha(pv_{j+1}^* + qv_{j-1}^*) > f_j, \qquad\qquad 0 < j < j^*,$$
$$v_j^* = f_j \qquad\qquad\qquad > \alpha(pv_{j+1}^* + qv_{j-1}^*), \qquad j^* \leq j.$$

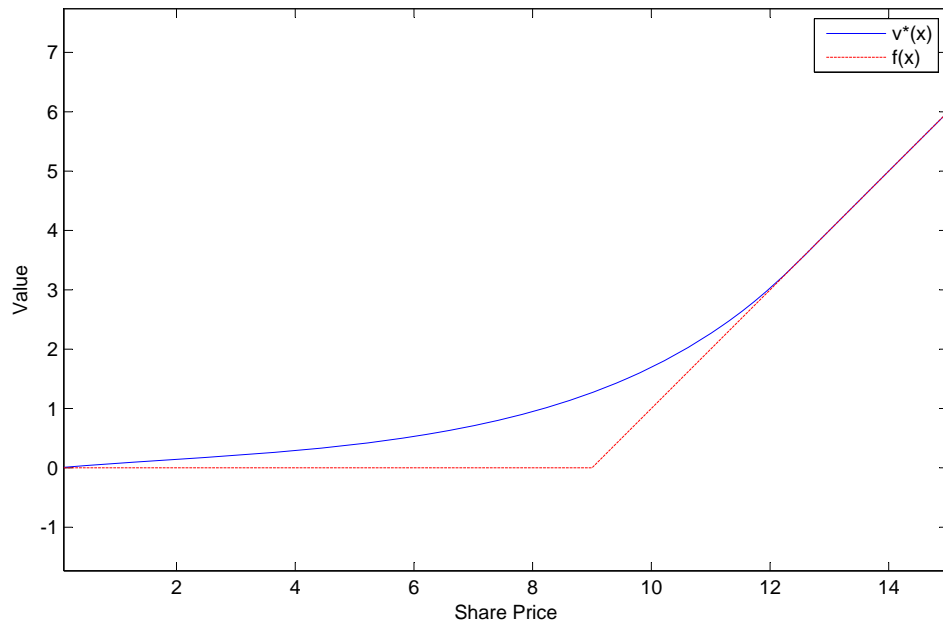Prove this statement. If you need additional assumptions, state them clearly and discuss their implication.

FIGURE 1. Plot of $v^*(x)$ and $f(x)$ corresponding to the case where $\alpha = 0.999$, $p = 0.51$, $\Delta x = 0.1$, and $S = 9$. In this case, $x_{j^*} = 12.4$.