Optimization Qualifying Exam

Solution

September 2007

1. Primal Problem.

The primal problem is:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{j=0}^{\infty} v_j & \\
\text{subject to} & v_j \geq f_j, & j \geq 0, \\
 & v_j \geq \alpha\left(pv_{j+1} + qv_{j-1}\right), & j \geq 1.
\end{array}
$$

2. Dual Problem.

The associated dual problem is

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=0}^{\infty} f_j y_j & \\
\text{subject to} & y_0 - \alpha q z_1 = 1, & \\
 & y_1 + z_1 - \alpha q z_2 = 1, & \\
 & y_j - \alpha p z_{j-1} + z_j - \alpha q z_{j+1} = 1, & j \geq 2, \\
 & y_j \geq 0, & j \geq 0, \\
 & z_j \geq 0, & j \geq 1.
\end{array}
$$

3. Statement of Claim.

Let v_j denote the optimal primal solution and y_j and z_j the optimal dual solution (i.e., we are dropping the usual "stars" that denote optimality). Suppose, as claimed, that there exists a j^* such that

$$
\begin{array}{lllll}
v_0 & = & f_0, & & \\
v_j & = & \alpha(pv_{j+1} + qv_{j-1}) & > \quad f_j, & \text{for } 0 < j < j^*, \\
v_j & = & f_j & > \quad \alpha(pv_{j+1} + qv_{j-1}), & \text{for } j^* \leq j.
\end{array}
$$

4. Invoke Complementarity.

Complementarity implies that

$$\begin{aligned} z_j &= 0, \qquad j \geq j^*, & (1) \\ y_j &= 0, \qquad 0 < j < j^*. & (2) \end{aligned}$$

Dual feasibility with (1) implies

$$\begin{aligned} y_{j^*} - \alpha p z_{j^*-1} &= 1, \\ y_j &= 1, \quad j > j^*. \end{aligned}$$

Dual feasibility with (2) implies

$$\begin{aligned} z_1 - \alpha q z_2 &= 1, \\ -\alpha p z_{j-1} + z_j - \alpha q z_{j+1} &= 1, \quad 1 < j < j^*. \end{aligned}$$

5. Second Order Difference Equations.

Hence, the problem of solving the equalities has been reduced to a pair of second order difference equations with Dirichlet boundary conditions. The first difference equation is

$$\begin{aligned} v_j - \alpha(p v_{j+1} + q v_{j-1}) &= 0, \qquad 0 < j < j^*, & (3) \\ v_0 &= 0, & (4) \\ v_{j^*} &= f_{j^*} & (5) \end{aligned}$$

and the second one is

$$\begin{aligned} z_j - \alpha(p z_{j-1} + q z_{j+1}) &= 1, \qquad 0 < j < j^*, & (6) \\ z_0 &= 0, & (7) \\ z_{j^*} &= 0. & (8) \end{aligned}$$

Note that in (4) we used the fact that $f_0 = 0$ and in (6) we have added a new variable, z_0, which is just fixed to zero (by (7)). In this way we consolidate the difference equation for z_j to a more elegant form.

6. Explicit Solution of Difference Equation for v_j.

First, we solve the equation for v_j. To this end, suppose that

$$v_j = \xi^j$$

for some positive real number ξ. Substituting into the difference equation, we get

$$\xi^j - \alpha(p\xi^{j+1} + q\xi^{j-1}) = 0.$$

Dividing by ξ^{j-1}, we get a quadratic equation

$$-\alpha p \xi^2 + \xi - \alpha q = 0.$$

The two roots of this equation are

$$\xi_\pm = \frac{-1 \pm \sqrt{1-4\alpha^2 pq}}{-2\alpha p}.$$

The general solution to the difference equation is therefore

$$v_j = c_+ \xi_+^j + c_- \xi_-^j.$$

From the first boundary condition (4), we get that $c_- = -c_+$. This relation together with the second boundary condition (5) gives

$$c_+ = \frac{f_{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}}.$$

Hence,

$$v_j = f_{j^*} \frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}, \qquad 0 < j < j^*. \tag{9}$$

7. Explicit Solution of Difference Equation for z_j.

Now, let's solve for z_j. We need a particular solution to the equation and the general solution to the associated homogeneous equation. For a particular solution, we try the simplest thing

$$z_j \equiv c.$$

Substituting into the difference equation, we discover that $c = 1/(1-\alpha)$. The associated homogeneous equation is exactly the same as the equation for v_j except with p and q interchanged. Hence the general solution, which is the sum of the particular and the homogeneous, is given by

$$z_j = \frac{1}{1-\alpha} + c_+ \zeta_+^j + c_- \zeta_-^j$$

where

$$\begin{aligned} \zeta_+ &= 1/\xi_- = \frac{-1 + \sqrt{1-4\alpha^2 pq}}{-2\alpha q}, \\ \zeta_- &= 1/\xi_+ = \frac{-1 - \sqrt{1-4\alpha^2 pq}}{-2\alpha q}. \end{aligned}$$

Using the boundary conditions to eliminate the two undetermined constants, we get

$$z_j = \left(1 - \frac{\zeta_-^{j^*} - 1}{\zeta_-^{j^*} - \zeta_+^{j^*}} \zeta_+^j - \frac{\zeta_+^{j^*} - 1}{\zeta_+^{j^*} - \zeta_-^{j^*}} \zeta_-^j \right) \Big/ (1-\alpha), \qquad 0 < j < j^*.$$

8. Solution to Equalities.

To summarize, we have

$$v_j = \begin{cases} 0 & j = 0 \\ f_{j^*} \frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}, & 0 < j < j^* \\ f_j & j^* \leq j \end{cases}$$

$$z_j = \begin{cases} \left(1 - \frac{\zeta_-^{j^*}-1}{\zeta_-^{j^*} - \zeta_+^{j^*}} \zeta_+^j - \frac{\zeta_+^{j^*}-1}{\zeta_+^{j^*} - \zeta_-^{j^*}} \zeta_-^j \right) \Big/ (1-\alpha) & 0 < j < j^* \\ 0 & j^* \leq j \end{cases}$$

$$y_j = \begin{cases} 1 + \alpha q z_1 & j = 0 \\ 0 & 0 < j < j^* \\ 1 + \alpha p z_{j^*-1} & j = j^* \\ 1 & j^* < j \end{cases}$$

9. Check the Inequalities.

All that remains is to show that the various inequalities are satisfied:

$$y_j \geq 0, \qquad j \geq 0, \tag{10}$$
$$z_j \geq 0, \qquad j \geq 1, \tag{11}$$
$$v_j \geq f_j, \qquad j \geq 0, \tag{12}$$
$$v_j \geq \alpha(pv_{j+1} + qv_{j-1}), \qquad j \geq 1. \tag{13}$$

9.1. **Inequalities** (11)**.** Inequalities (11) follow trivially for $j \geq j^*$ from the formula given above for z_j. To check them for $j < j^*$, we do a proof by contradiction. So, suppose that $z_j < 0$ for some $0 < j < j^*$. Then there exists a k at which z_k is negative and a local minimum:

$$z_k < z_{k-1} \quad \text{and} \quad z_k < z_{k+1}.$$

But, we also have

$$\begin{aligned} z_k &= 1 + \alpha(pz_{k-1} + qz_{k+1}) \\ &> 1 + \alpha(pz_k + qz_k) \\ &= 1 + \alpha z_k. \end{aligned}$$

Rearranging, we get $z_k > 1/(1-\alpha) > 0$, which contradicts the assumption that z_k is negative. Hence, inequalities (11) hold for all j. (This is a simple example of a *minimum principle* as one encounters in harmonic analysis.)

9.2. **Inequalities** (10)**.** These follow trivially from inequalities (11) and the formula for y_j.

9.3. **Inequalities** (13)**.** These hold trivially for $j < j^*$. They also hold trivially for $j > j^*$ **provided we assume that** $\alpha p \leq 1/2$ and $\alpha q \leq 1/2$. We'll come back to the inequality (13) for $j = j^*$ after we consider inequalities (12).

9.4. **Inequalities** (12)**.** For $j \geq j^*$, these are trivial. Furthermore, it follows immediately from (9) that $v_j \geq 0$ for all j. Hence, we just need to check that $v_j \geq x_j - S$ for $j < j^*$. In order to have these inequalities hold for $j < j^*$, **we need to pick**

$$j^* \in K := \left\{ k : f_k \, \frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^k - \xi_-^k} > f_{k-1} \right\}.$$

Of course, we need to assume that K is nonempty. Clearly no k for which $x_k < S$ can belong to the set (because both f_k and f_{k-1} vanish). For convenience, then, we assume that $S \in E$—that is, $S = j_S \Delta x$ for some j_S. In that case, **we assume that** $k = j_S + 1$ **belongs to the set:**

$$f_{j_S+1} \, \frac{\xi_+^{j_S} - \xi_-^{j_S}}{\xi_+^{j_S+1} - \xi_-^{j_S+1}} > f_{j_S}.$$

Let $h_j = x_j - S$. With such a choice and the assumption that $j^* \in K$, we have that $v_{j^*} = h_{j^*}$ and $v_{j^*-1} > h_{j^*-1}$. Suppose that $v_{j'} < h_{j'}$ for some $j' < j^*$. Then the sequence $u_j := v_j - h_j$ must have a local maximum at some point, say k, strictly between j' and j^*. That is, $u_k > u_{k-1}$ and $u_k > u_{k+1}$. But, we also have

$$\begin{aligned} u_k &= v_k - h_k \\ &= \alpha(p v_{k+1} + q v_{k-1}) - \frac{1}{2}(h_{k+1} + h_{k-1}) \\ &\leq \frac{1}{2}(v_{k+1} + v_{k-1}) - \frac{1}{2}(h_{k+1} + h_{k-1}) \\ &= \frac{1}{2}(u_{k+1} + u_{k-1}) \\ &< u_k. \end{aligned}$$

Clearly this is impossible. Hence, u_j can't have a local maximum and therefore v_j cannot dip below h_j.

9.5. **Inequality** (13) **with** $j = j^*$**.** Finally, to get inequality (13) for $j = j^*$, we need to assume that $j^* + 1 \notin K$. That is,

$$f_{j^*+1} \, \frac{\xi_+^{j^*} - \xi_-^{j^*}}{\xi_+^{j^*+1} - \xi_-^{j^*+1}} \leq f_{j^*}. \tag{14}$$

To see why, let w_j denote the solution to the difference equation

$$\begin{aligned} w_j - \alpha(p w_{j+1} + q w_{j-1}) &= 0, \qquad 0 < j, \\ w_0 &= 0, \\ w_{j^*} &= f_{j^*}. \end{aligned}$$

This is the same as (6)–(8) but extended to all j. Clearly we have $v_{j^*} = w_{j^*}$ and $v_{j^*-1} = w_{j^*-1}$. Hence, (13) at j^* will hold if and only if $v_{j^*+1} \leq w_{j^*+1}$:

$$f_{j^*+1} = v_{j^*+1} \leq w_{j^*+1} = f_{j^*} \, \frac{\xi_+^{j^*+1} - \xi_-^{j^*+1}}{\xi_+^{j^*} - \xi_-^{j^*}}$$

The resulting inequality is clearly equivalent to (14).

10. Summary

The claimed result holds provided we make the extra assumptions that

(1) $\alpha p \leq 1/2$ and $\alpha q \leq 1/2$,
(2) $S = j_S \Delta x$ for some integer j_S, and
(3) $f_{j_S+1} \left(\xi_+^{j_S} - \xi_-^{j_S}\right) / \left(\xi_+^{j_S+1} - \xi_-^{j_S+1}\right) > f_{j_S}$.

With these assumptions, j^* must be chosen as

$$j^* := \max\left\{k : f_k \, \frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^{k} - \xi_-^{k}} > f_{k-1}\right\}.$$