ORF 522: Lecture 17

Extreme Optics
and
The Search for Earth-Like Planets

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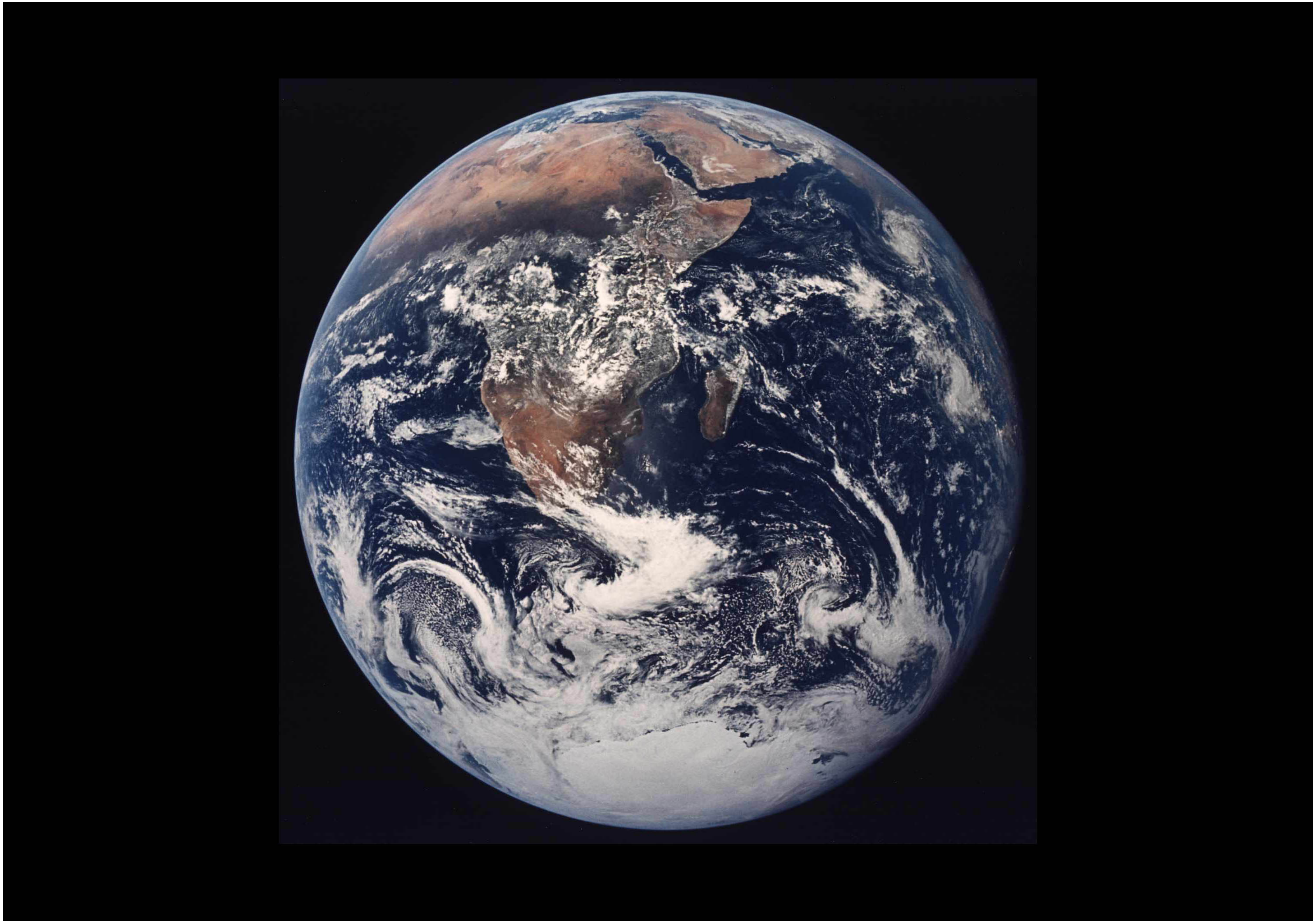
ABSTRACT

- NASA/JPL plans to build and launch a space telescope to look for Earth-like planets.
- I will describe the detection problem and explain why it is hard.
- Optimization is key to several design concepts.

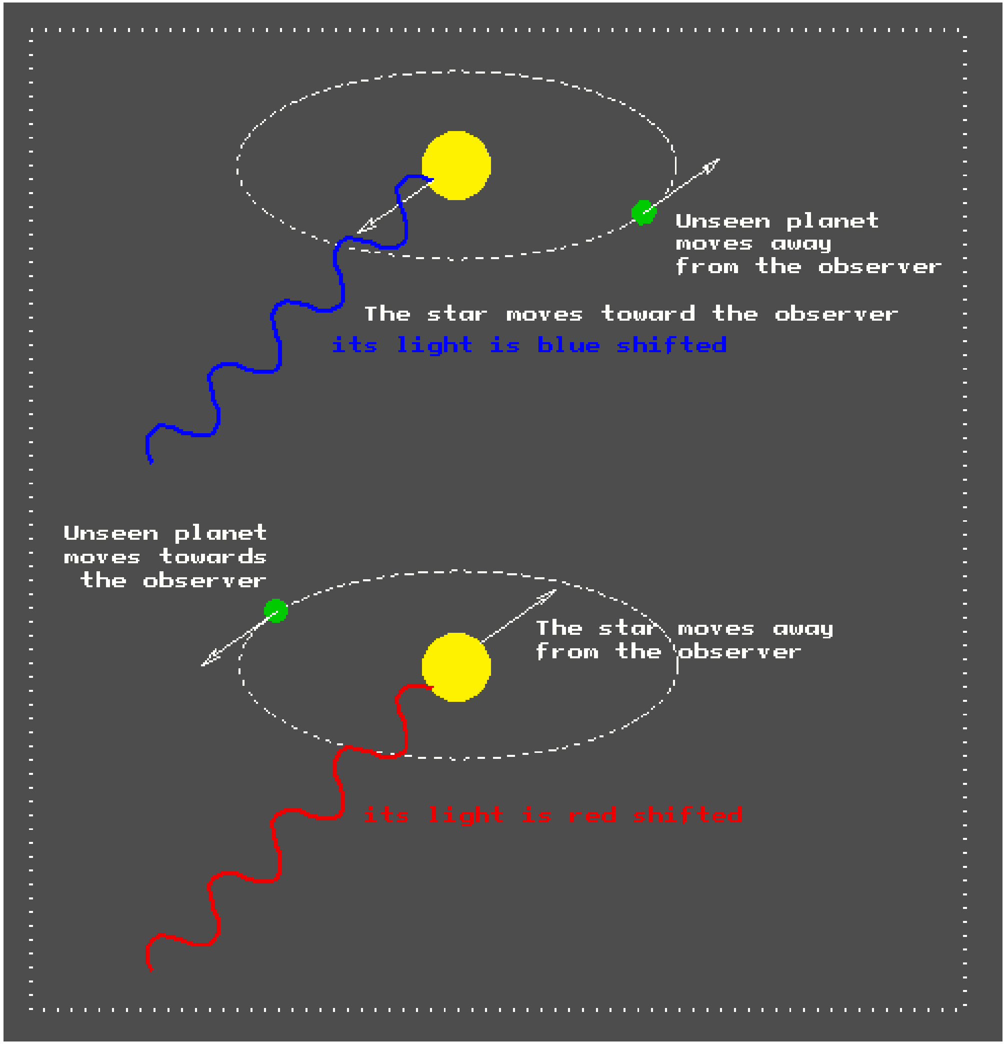
Are We Alone?



Wobble Methods

Radial Velocity.
For edge-on systems.
Measure periodic doppler shift.

Astrometry.
Best for face-on systems.
Measure circular wobble against background stars.

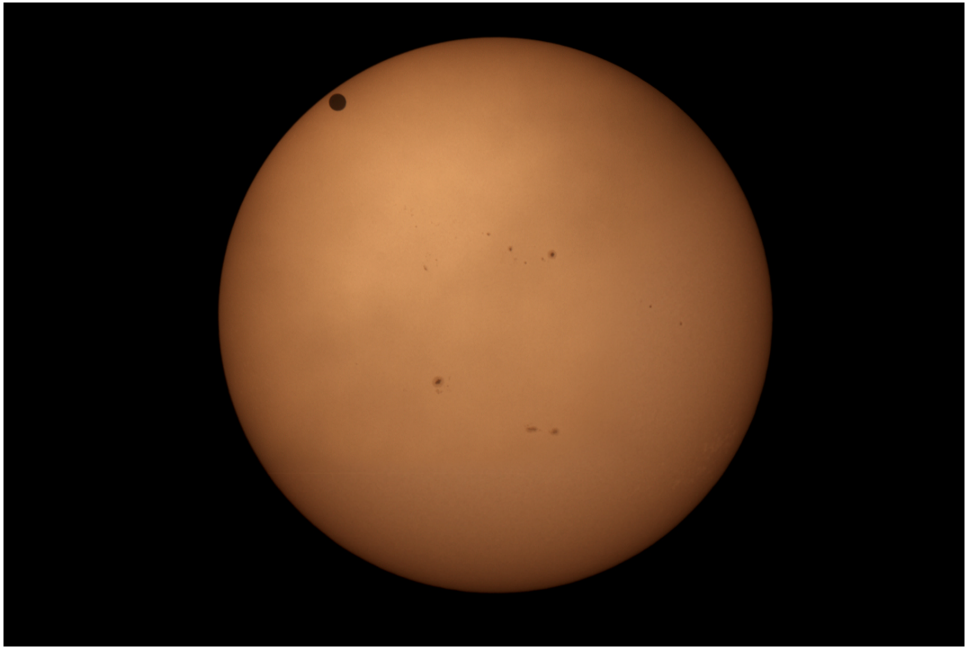


The Transit Method

A few planets have been discovered using the **Transit Method**.

On June 6, 2012, Venus transited in front of the Sun.

I took a picture of this event with my small telescope.



If we on Earth are lucky to be in the right position at the right time, we can detect similar transits of exosolar planets.

A few exosolar planets have been discovered this way.

Terrestrial Planet Finder Telescope

- NASA/JPL space telescope.

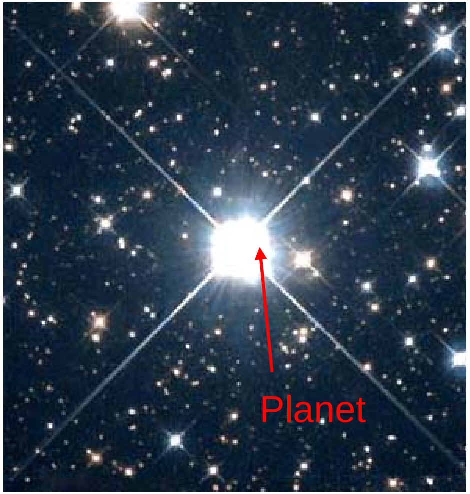
- Launch date: ~~2014~~...well, sometime in our lifetime.

- DETECT: Search 150-500 nearby (5-15 pc distant) Sun-like stars for Earth-like planets.

- CHARACTERIZE: Determine basic physical properties and measure "biomarkers", indicators of life or conditions suitable to support it.

Why Is It Hard?

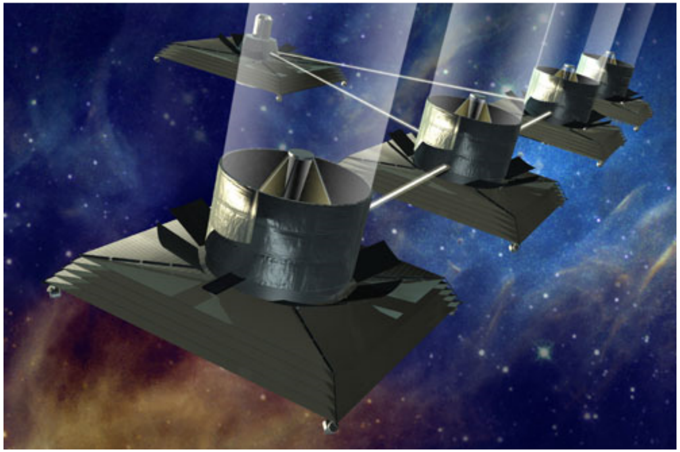
- **Contrast.** Star $= 10^{10} \times$ Planet
- **Angular Separation.** 0.1 arcseconds.



Early Design Concepts

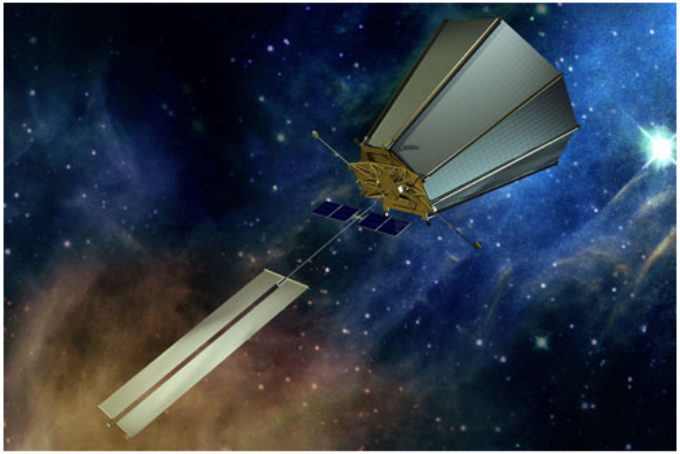
Space-based infrared nulling interferometer (TPF-I).

TPF-Interferometer



Visible-light telescope with an elliptical mirror (3.5 m x 8 m) and an **optimized** diffraction control system (TPF-C).

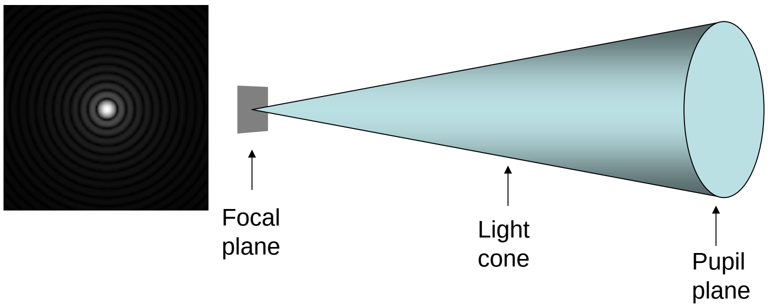
TPF-Coronagraph



Diffraction Control via Shaped Pupils

Consider a telescope. Light enters the front of the telescope—the **pupil plane**.

The telescope focuses the light passing through the pupil plane from a given direction at a certain point on the **focal plane**, say $(0,0)$.



However, a point source produces not a point image but an **Airy pattern** consisting of an **Airy disk** surrounded by a system of **diffraction rings**.

These diffraction rings are too bright. The rings would completely hide the planet.

By placing a mask over the pupil, one can control the shape and strength of the diffraction rings. The problem is to find an optimal shape so as to put a very deep **null** very close to the Airy disk.

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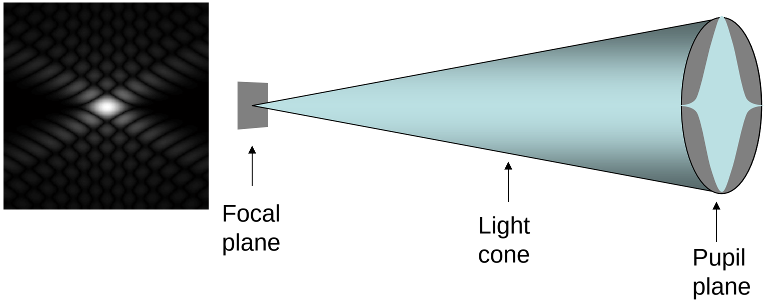

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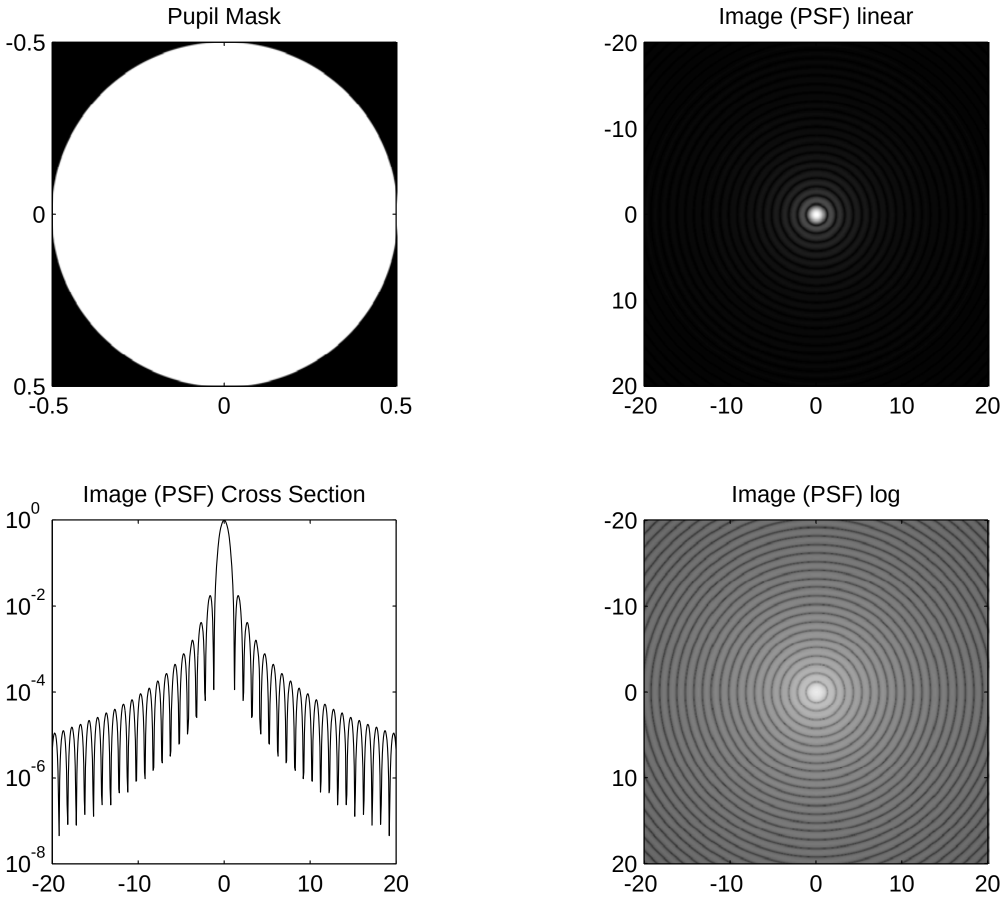
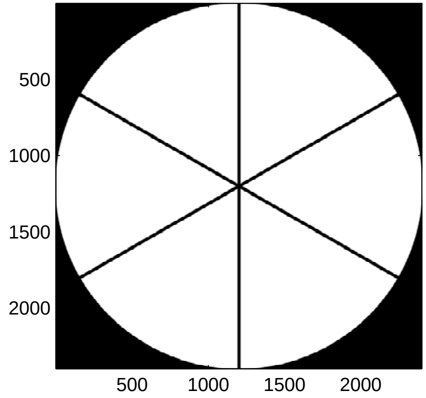
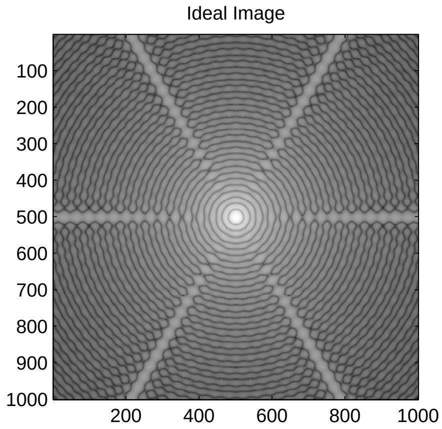
The Airy Pattern



Spiders are an Example of a Shaped Pupil





Note the six bright radial **spikes**

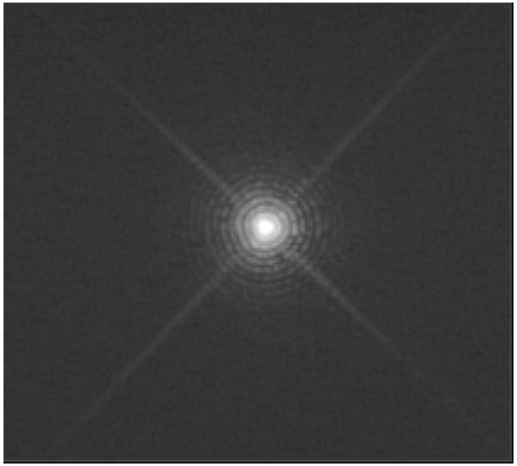
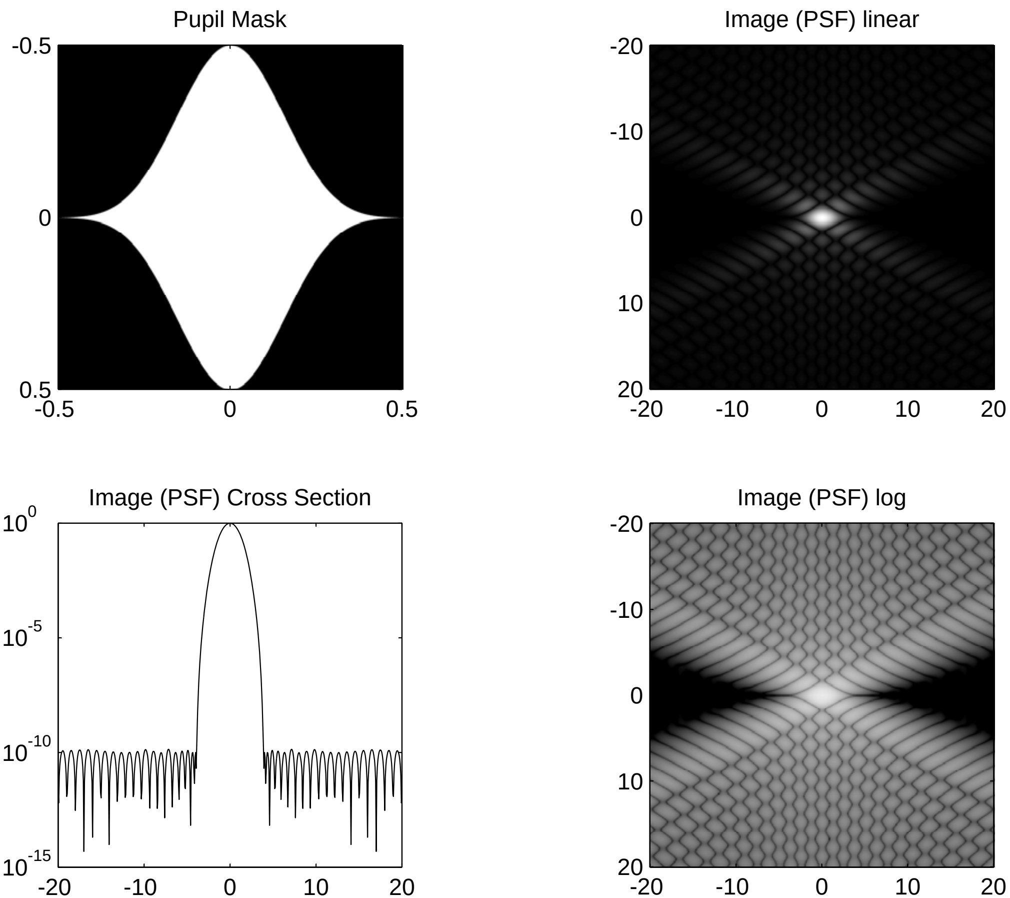


Image of Vega taken with my "big" 250mm telescope.

The Seven Sisters with Spikes

Pleiades image taken with small refractor equipped with **dental floss** spiders.

The Spergel-Kasdin-Vanderbei Pupil

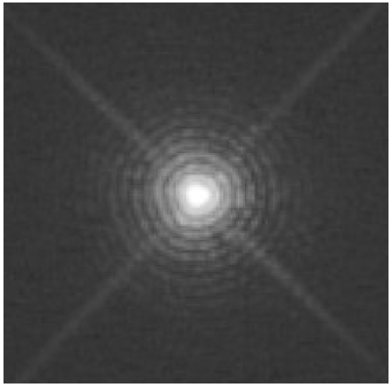


High-Contrast Imaging for Planet-Finding

Build a telescope capable of finding Earth-like planets around nearby Sun-like stars.

Problem is hard:

- Star is 10^{10} times brighter than the planet.
- Angular separation is small ≈ 0.1 arcseconds.
- Light is a wave: the star is not a pinpoint of light—it has a diffraction pattern.
- Light is photons $\Rightarrow$ Poisson statistics.



The diffraction pattern is the magnitude-squared of the Fourier transform of the telescope's pupil.

Pupil Apodization

Let $f(x, y)$ denote the transmissivity (i.e., apodization) at location (x, y) on the surface of a filter placed in the pupil of a telescope.

The electromagnetic field in the image plane of such a telescope associated with an on-axis point source (i.e., a star) is proportional to the Fourier transform of the apodization f.

Assuming that the telescope's opening has a radius of one, the Fourier transform can be written as

$$\widehat{f}(\xi, \eta) = \iint_{\square} e^{2\pi i(x\xi + y\eta)} f(x, y) dx dy.$$

The intensity of the light in the image is proportional to the magnitude squared of $\widehat{f}$.

Assuming that the underlying telescope has a circular opening of radius one, we impose the following constraint on f:

$$f(x, y) = 0 \quad \text{for} \quad x^2 + y^2 > 1.$$

Optimized Apodizations

Maximize light throughput subject to constraint that almost no light reaches a given dark zone $\mathcal{D}$ and other structural constraints:

$$
\begin{array}{lrll}
\text{maximize} & \displaystyle\iint_{\square} f(x,y)\,dx\,dy & \left(\quad = \widehat{f}(0,0) \quad \right) & \\
\text{subject to} & \left|\widehat{f}(\xi,\eta)\right| & \leq \varepsilon\, \widehat{f}(0,0), & (\xi,\eta) \in \mathcal{D}, \\
 & f(x,y) & = 0, & x^2+y^2 > 1, \\
 & 0 \leq f(x,y) & \leq 1, & \text{for all } x, y.
\end{array}
$$

Here, ε is a small positive constant (on the order of 10^{-5}).

In general, the Fourier transform $\widehat{f}$ is complex valued.

This optimization problem has a linear objective function and both linear constraints and second-order cone constraints.

Hence, a discretized version can be solved (to a global optimum).

Exploiting Symmetry

Assuming that the filter can be symmetric with respect to reflection about both axes (note: sometimes not possible), the Fourier transform can be written as

$$\widehat{f}(\xi,\eta) = 4\int_0^1\int_0^1 \cos(2\pi x\xi)\cos(2\pi y\eta) f(x,y)\,dx\,dy.$$

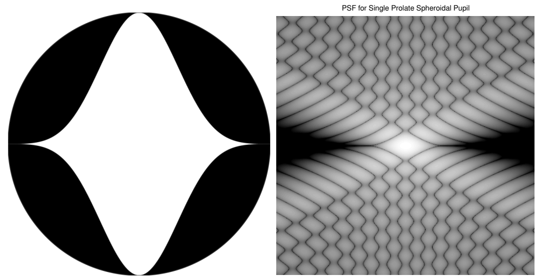
In this case, the Fourier transform is real and so the second-order cone constraints can be replaced with a pair of inequalities,

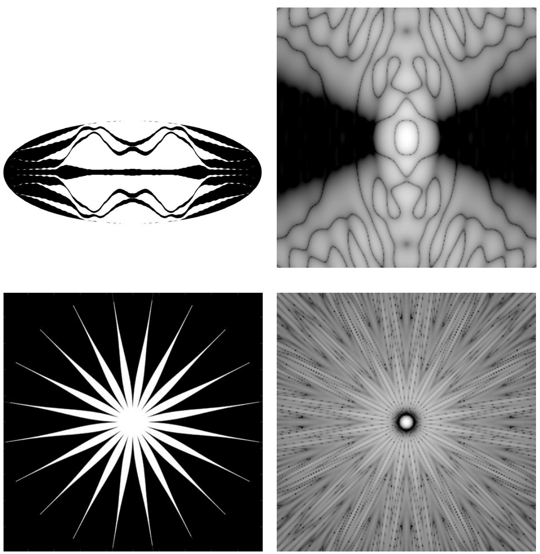
$$-\varepsilon\,\widehat{f}(0,0) \le \widehat{f}(\xi,\eta) \le \varepsilon\,\widehat{f}(0,0),$$

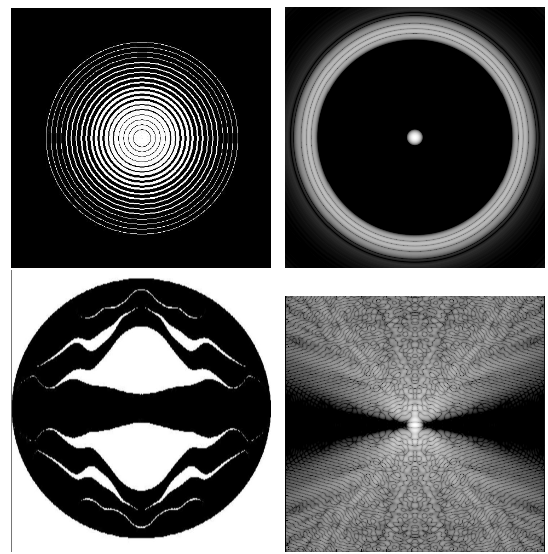
making the problem an infinite dimensional linear programming problem.

Curse of Dimensionality: $2 > 1$.

Potpourri of Pupil Masks







Discretization

Consider a two-dimensional Fourier transform

$$\widehat{f}(\xi,\eta) = 4\int_0^1\int_0^1 \cos(2\pi x\xi)\cos(2\pi y\eta) f(x,y)\,dx\,dy.$$

Its discrete approximation can be computed as

$$\widehat{f}_{j_1,j_2} = 4\sum_{k_2=1}^{n}\sum_{k_1=1}^{n}\cos(2\pi x_{k_1}\xi_{j_1})\cos(2\pi y_{k_2}\eta_{j_2}) f_{k_1,k_2}\Delta x\Delta y, \quad 1 \le j_1, j_2 \le m,$$

where

$$\begin{aligned}
\begin{bmatrix} x_k \\ y_k \end{bmatrix} &= (k-1/2)\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \qquad && 1 \le k \le n,\\
\begin{bmatrix} \xi_j \\ \eta_j \end{bmatrix} &= (j-1/2)\begin{bmatrix} \Delta \xi \\ \Delta \eta \end{bmatrix}, && 1 \le j \le m,\\
f_{k_1,k_2} &= f(x_{k_1}, y_{k_2}), && 1 \le k_1, k_2 \le n,\\
\widehat{f}_{j_1,j_2} &\approx \widehat{f}(\xi_{j_1}, \eta_{j_2}), && 1 \le j_1, j_2 \le m.
\end{aligned}$$

Complexity: $m^2 n^2$.

A Clever (and Trivial!) Idea

The obvious brute force calculation requires $m^2 n^2$ operations.

However, we can “factor” the double sum into a nested pair of sums.

Introducing new variables that represent the inner sum, we get:

$$g_{j_1,k_2} = 2 \sum_{k_1=1}^{n} \cos(2\pi x_{k_1} \xi_{j_1}) f_{k_1,k_2} \Delta x, \qquad 1 \le j_1 \le m, \;\; 1 \le k_2 \le n,$$

$$\widehat{f}_{j_1,j_2} = 2 \sum_{k_2=1}^{n} \cos(2\pi y_{k_2} \eta_{j_2}) g_{j_1,k_2} \Delta y, \qquad 1 \le j_1, j_2 \le m,$$

Formulated this way, the calculation requires only $mn^2 + m^2 n$ operations.

Brute Force vs Clever Approach

On the following page we show two AMPL model formulations of this problem.

On the left is the version expressed in the straightforward one-step manner.

On the right is the AMPL model for the same problem but with the Fourier transform expressed as a pair of transforms—the so-called two-step process.

The dark zone $\mathcal{D}$ is a pair of sectors of an annulus with inner radius 4 and outer radius 20.

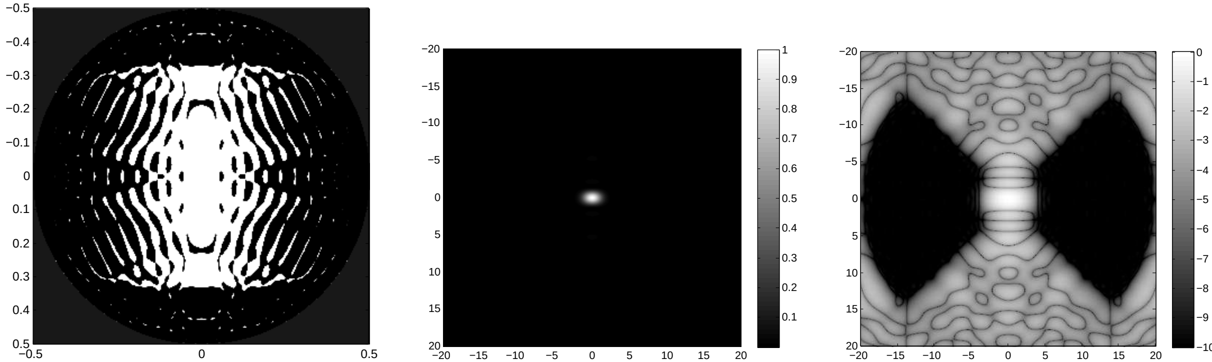
Except for the resolution, the two models produce the same result.

Two AMPL Models

```
param rho0 := 4;        param rho1 := 20;
param m := 35;      # discretization parameter
param n := 150;     # discretization parameter
param dx := 1/(2*n);    param dy := dx;

set Xs := setof {j in 0.5..n-0.5 by 1} j/(2*n);
set Ys := Xs;
set Pupil :=
    setof {x in Xs, y in Ys: x^2+y^2<0.25} (x,y);
set Xis  := setof {j in 0..m} j*rho1/m;
set Etas := Xis;
set DarkHole := setof {xi in Xis, eta in Etas:
                        xi^2+eta^2>=rho0^2 &&
                        xi^2+eta^2<=rho1^2 &&
                        eta <= xi } (xi,eta);

var f {(x,y) in Pupil} >= 0, <= 1;

var fhat {xi in Xis, eta in Etas};

maximize area: sum {(x,y) in Pupil} f[x,y]*dx*dy;

subject to fhat_def {xi in Xis, eta in Etas}:
    fhat[xi,eta] = 4*sum {(x,y) in Pupil}
        f[x,y]*cos(2*pi*x*xi)
              *cos(2*pi*y*eta)*dx*dy;
subject to sidelobe_pos {(xi,eta) in DarkHole}:
    fhat[xi,eta] <= 10^(-5)*fhat[0,0];
subject to sidelobe_neg {(xi,eta) in DarkHole}:
    -10^(-5)*fhat[0,0] <= fhat[xi,eta];

solve;
```

```
param rho0 := 4;        param rho1 := 20;
param m := 35;      # discretization parameter
param n := 1000;    # discretization parameter
param dx := 1/(2*n);    param dy := dx;

set Xs := setof {j in 0.5..n-0.5 by 1} j/(2*n);
set Ys := Xs;
set Pupil :=
    setof {x in Xs, y in Ys: x^2+y^2 < 0.25} (x,y);
set Xis  := setof {j in 0..m} j*rho1/m;
set Etas := Xis;
set DarkHole := setof {xi in Xis, eta in Etas:
                        xi^2+eta^2>=rho0^2 &&
                        xi^2+eta^2<=rho1^2 &&
                        eta <= xi } (xi,eta);

var f {(x,y) in Pupil} >= 0, <= 1;
var g {xi in Xis, y in Ys};
var fhat {xi in Xis, eta in Etas};

maximize area: sum {(x,y) in Pupil} f[x,y]*dx*dy;

subject to g_def {xi in Xis, y in Ys}:
    g[xi,y] = 2*sum {x in Xs: (x,y) in Pupil}
        f[x,y]*cos(2*pi*x*xi)*dx;
subject to fhat_def {xi in Xis, eta in Etas}:
    fhat[xi,eta] = 2*sum {y in Ys}
        g[xi,y]*cos(2*pi*y*eta)*dy;

subject to sidelobe_pos {(xi,eta) in DarkHole}:
    fhat[xi,eta] <= 10^(-5)*fhat[0,0];
subject to sidelobe_neg {(xi,eta) in DarkHole}:
    -10^(-5)*fhat[0,0] <= fhat[xi,eta];

solve;
```

Optimal Solution



Left. The optimal apodization found by either of the models shown on previous slide.

Center. Plot of the star's image (using a linear stretch).

Right. Logarithmic plot of the star's image (black $= 10^{-10}$).

Notes:

- The "apodization" turns out to be purely opaque and transparent (i.e., a mask).
- The mask has "islands" and therefore must be laid on glass.

Close Up

Brute force with $n = 150$

Two-step with $n = 1000$

Summary Problem Stats

Comparison between a few sizes of the one-step and two-step models.

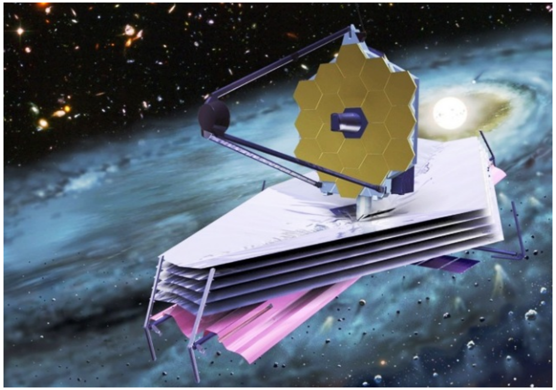
Problem-specific stats.

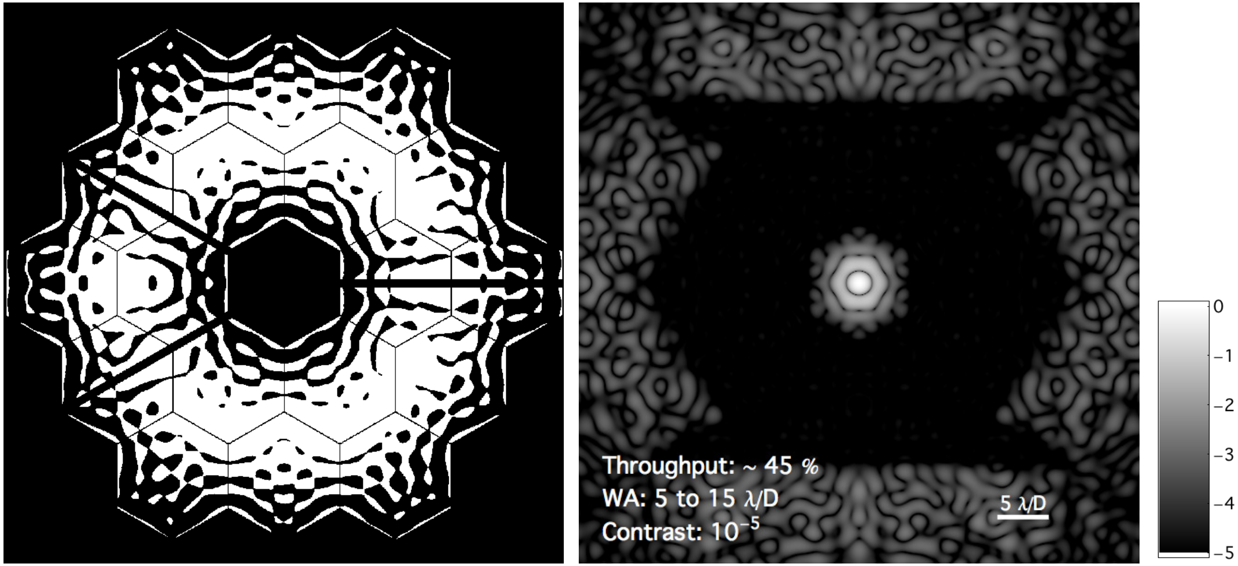
Model	n	m	constraints	variables	nonzeros	arith. ops.
One step	150	35	976	17,672	17,247,872	17,196,541,336
One step	250	35	*	*	*	*
Two step	150	35	7,672	24,368	839,240	3,972,909,664
Two step	500	35	20,272	215,660	7,738,352	11,854,305,444
Two step	1000	35	38,272	822,715	29,610,332	23,532,807,719

Hardware/Solution-specific performance comparison data.

Model	n	m	iterations	primal objective	dual objective	cpu time (sec)
One step	150	35	54	0.05374227247	0.05374228041	1380
One step	250	35	*	*	*	*
Two step	150	35	185	0.05374233071	0.05374236091	1064
Two step	500	35	187	0.05395622255	0.05395623990	4922
Two step	1000	35	444	0.05394366337	0.05394369256	26060

JWST





Repurposed NRO Spy Satellite