# ORF 522: Lecture 19

# Nonlinear Optimization: Models

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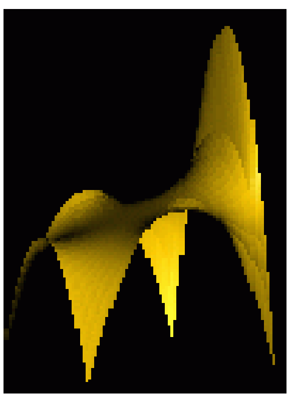
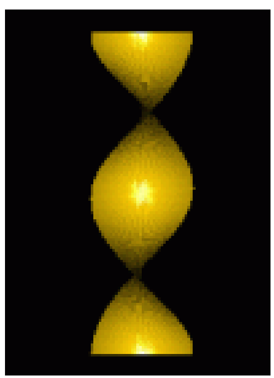
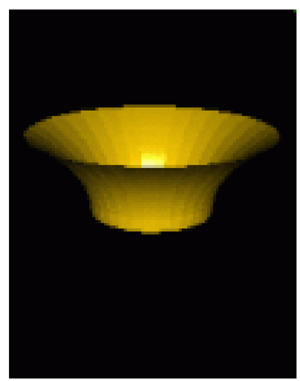
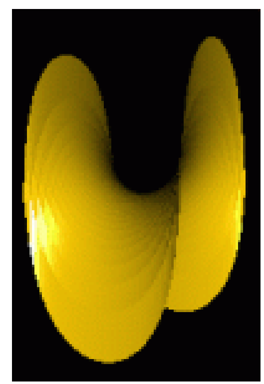
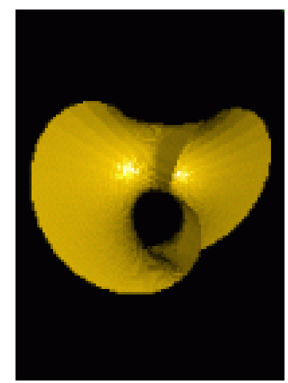
# Examples: Convex Optimization Models

# Minimal Surfaces

- Given: a domain $D$ in $R^2$ and an embedding $\mathbf{x} = (x_1, x_2, x_3)$ of its boundary $\partial D$ in $R^3$;
- Find: an embedding of the entire domain into $R^3$ that is consistent with the boundary embedding and has minimal surface area:

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\iint_D \left\| \frac{\partial \mathbf{x}}{\partial s} \times \frac{\partial \mathbf{x}}{\partial t} \right\| ds dt \\
\text{subject to} & \mathbf{x}(s,t) \text{ fixed for } (s,t) \in \partial D \\
 & x_1(s,t) \text{ fixed for } (s,t) \in D \\
 & x_2(s,t) \text{ fixed for } (s,t) \in D
\end{array}
$$

The specific problems coded below take $D$ to be either a square or an annulus.

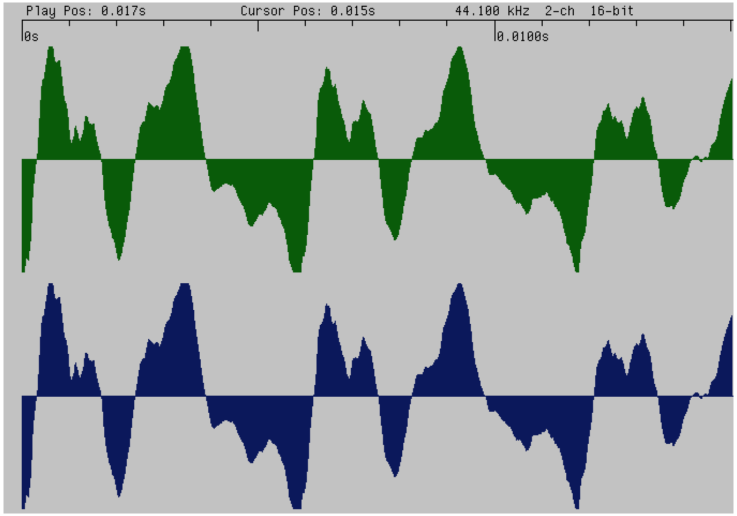
# Specific Example

Scherk.mod with $D$ discretized into a $64 \times 64$ grid gives the following results:

| | |
|---|---|
| constraints | 0 |
| variables | 3844 |
| time (secs) | |
| LOQO | 5.1 |
| LANCELOT | 4.0 |
| SNOPT | * |

# Finite Impulse Response (FIR) Filter Design

- Audio is stored digitally in a computer as a stream of short integers: $u_k,\ k \in \mathbb{Z}$.
- When the music is played, these integers are used to drive the displacement of the speaker from its resting position.
- For CD quality sound, 44100 short integers get played per second per channel.




| | | | | | |
|---|---|---|---|---|---|
| 0 | -32768 | 8 | -23681 | 16 | 12111 |
| 1 | -32768 | 9 | -18449 | 17 | 17311 |
| 2 | -32768 | 10 | -11025 | 18 | 21311 |
| 3 | -30753 | 11 | -6913 | 19 | 23055 |
| 4 | -28865 | 12 | -4337 | 20 | 23519 |
| 5 | -29105 | 13 | -1329 | 21 | 25247 |
| 6 | -29201 | 14 | 1743 | 22 | 27535 |
| 7 | -26513 | 15 | 6223 | 23 | 29471 |

# FIR Filter Design—Continued

- A *finite impulse response (FIR) filter* takes as input a digital signal and convolves this signal with a finite set of fixed numbers $h_{-n}, \dots, h_n$ to produce a filtered output signal:

$$y_k = \sum_{i=-n}^{n} h_i u_{k-i}.$$

- Sparing the details, the output power at frequency $\nu$ is given by

$$|H(\nu)|$$

where

$$H(\nu) = \sum_{k=-n}^{n} h_k e^{2\pi i k \nu},$$

- Similarly, the mean squared deviation from a flat frequency response over a frequency range, say $\mathcal{L} \subset [0,1]$, is given by

$$\frac{1}{|\mathcal{L}|} \int_{\mathcal{L}} |H(\nu) - 1|^2 \, d\nu$$

# Filter Design: Woofer, Midrange, Tweeter

minimize $\quad \rho$

subject to $\quad \displaystyle\int_0^1 \left(H_w(\nu) + H_m(\nu) + H_t(\nu) - 1\right)^2 d\nu \leq \epsilon$

$$\left(\frac{1}{|W|}\int_W H_w^2(\nu)d\nu\right)^{1/2} \leq \rho \qquad W = [.2, .8]$$

$$\left(\frac{1}{|M|}\int_M H_m^2(\nu)d\nu\right)^{1/2} \leq \rho \qquad M = [.4, .6] \cup [.9, .1]$$

$$\left(\frac{1}{|T|}\int_T H_t^2(\nu)d\nu\right)^{1/2} \leq \rho \qquad T = [.7, .3]$$

where

$$H_i(\nu) \;=\; h_i(0) + 2\sum_{k=1}^{n-1} h_i(k)\cos(2\pi k\nu), \quad i = W, M, T$$

$h_i(k) \;=\;$ filter coefficients, i.e., **decision variables**

# Specific Example

$$\begin{aligned} \text{filter length: } n &= 14 \\ \text{integral discretization: } N &= 1000 \end{aligned}$$

| | |
|---|---|
| constraints | 4 |
| variables | 43 |
| time (secs) | |
| LOQO | 79 |
| MINOS | 164 |
| LANCELOT | 3401 |
| SNOPT | 35 |

Ref: J.O. Coleman, U.S. Naval Research Laboratory,

CISS98 paper available:

Click *here* for demo

# Examples: Nonconvex Optimization Models

# Celestial Mechanics—Periodic Orbits

- Find periodic orbits for the planar gravitational $n$-body problem.
- Minimize action:

$$\int_0^{2\pi} (K(t) - P(t))dt,$$

- where $K(t)$ is kinetic energy,

$$K(t) = \frac{1}{2}\sum_i \left(\dot{x}_i^2(t) + \dot{y}_i^2(t)\right),$$

- and $P(t)$ is potential energy,

$$P(t) = -\sum_{i<j} \frac{1}{\sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2}}.$$

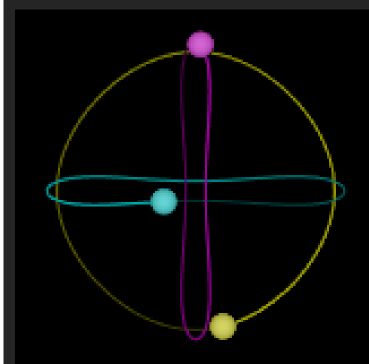
- Subject to periodicity constraints:

$$x_i(2\pi) = x_i(0), \qquad y_i(2\pi) = y_i(0).$$

# Specific Example

Orbits.mod with $n = 3$ and $(0, 2\pi)$ discretized into a $160$ pieces gives the following results:

| | | |
|---|---|---|
| constraints | 0 | |
| variables | 960 | |
| time (secs) | | |
| LOQO | 1.1 | |
| LANCELOT | 8.7 | |
| SNOPT | 287 | (no change for last 80% of iterations) |



Stable 3-Body Orbit

# Putting on an Uneven Green

Given:

- $z(x, y)$ elevation of the green.
- Starting position of the ball $(x_0, y_0)$.
- Position of hole $(x_f, y_f)$.
- Coefficient of friction $\mu$.

Find: initial velocity vector so that ball will roll to the hole and arrive with minimal speed.

Variables:

- $u(t) = (x(t), y(t), z(t))$—position as a function of time $t$.
- $v(t) = (v_x(t), v_y(t), v_z(t))$—velocity.
- $a(t) = (a_x(t), a_y(t), a_z(t))$—acceleration.
- $T$—time at which ball arrives at hole.

# Putting—Two Approaches

- Problem can be formulated with two decision variables:

$$v_x(0) \quad \text{and} \quad v_y(0)$$

  and two constraints:

$$x(T) = x_f \quad \text{and} \quad y(T) = y_f.$$

  In this case, $x(T)$, $y(T)$, and the objective function are complicated functions of the two variables that can only be computed by integrating the appropriate differential equation.

- A discretization of the complete trajectory (including position, velocity, and acceleration) can be taken as variables and the physical laws encoded in the differential equation can be written as constraints.

To implement the first approach, one would need an ode integrator that provides, in addition to the quantities being sought, first and possibly second derivatives of those quantities with respect to the decision variables.
The modern trend is to follow the second approach.

# Putting—Continued

Objective:

$$\text{minimize } v_x(T)^2 + v_y(T)^2.$$

Constraints:

$$\begin{aligned} v &= \dot{u} \\ a &= \dot{v} \\ ma &= N + F - mge_z \\ u(0) &= u_0 \qquad u(T) \;=\; u_f, \end{aligned}$$

where

- $m$ is the mass of the golf ball.
- $g$ is the acceleration due to gravity.
- $e_z$ is a unit vector in the positive $z$ direction.

and ...

# Putting—Continued

- $N = (N_x, N_y, N_z)$ is the normal force:

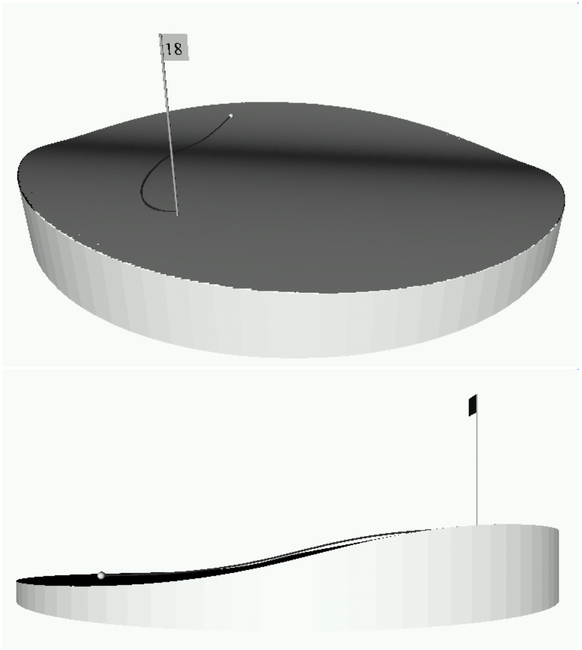
$$\begin{aligned} N_z &= m\frac{g - a_x(t)\frac{\partial z}{\partial x} - a_y(t)\frac{\partial z}{\partial y} + a_z(t)}{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1} \\ N_x &= -\frac{\partial z}{\partial x}N_z \\ N_y &= -\frac{\partial z}{\partial y}N_z. \end{aligned}$$

- $F$ is the force due to friction:

$$F = -\mu\|N\|\frac{v}{\|v\|}.$$

# Putting—Specific Example

- Discretize continuous time into $n = 200$ discrete time points.
- Use finite differences to approximate the derivatives.




| | |
|---|---|
| constraints | 597 |
| variables | 399 |
| time (secs) | |
| LOQO | 14.1 |
| LANCELOT | $> 600.0$ |
| SNOPT | 4.1 |

# Goddard Rocket Problem

Objective:

$$\text{maximize } h(T);$$

Constraints:

$$\begin{aligned}
v &= \dot{h} \\
a &= \dot{v} \\
\theta &= -c\dot{m} \\
ma &= (\theta - \sigma v^2 e^{-h/h_0}) - gm \\
0 &\leq \theta \leq \theta_{\max} \\
m &\geq m_{\min} \\
h(0) &= 0 \qquad v(0) = 0 \qquad m(0) = 3
\end{aligned}$$

where

- $\theta =$ Thrust , $m =$ mass
- $\theta_{\max}$, $g$, $\sigma$, $c$, and $h_0$ are given constants
- $h$, $v$, $a$, $T_h$, and $m$ are functions of time $0 \leq t \leq T$.

# Goddard Rocket Problem—Solution

| | |
|---|---|
| constraints | 399 |
| variables | 599 |
| time (secs) | |
| LOQO | 5.2 |
| LANCELOT | $(IL)$ |
| SNOPT | $(IL)$ |