ORF 522: Lecture 11

Linear Programming: Chapter 13
Network Flows: Algorithms

Robert J. Vanderbei

October 17, 2013

Slides last edited at 11:18am on Wednesday 16th April, 2014

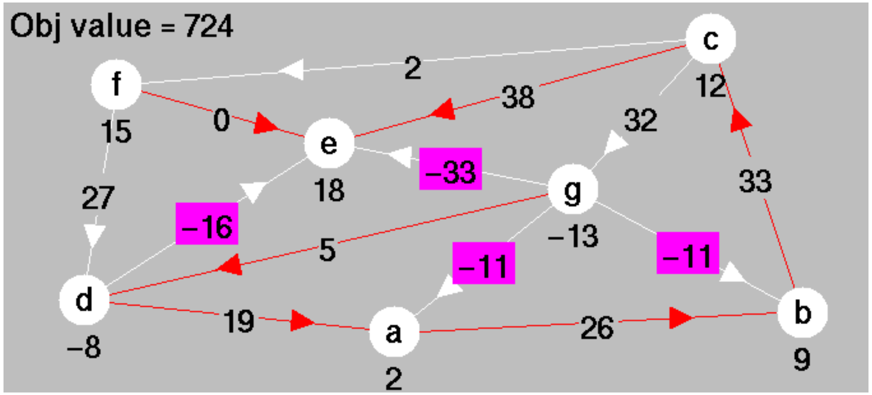
Operations Research and Financial Engineering, Princeton University
http://www.princeton.edu/∼rvdb

Agenda

- Primal Network Simplex Method
- Dual Network Simplex Method
- Two-Phase Network Simplex Method
- One-Phase Primal-Dual Network Simplex Method
- Planar Graphs
- Integrality Theorem

Primal Network Simplex Method

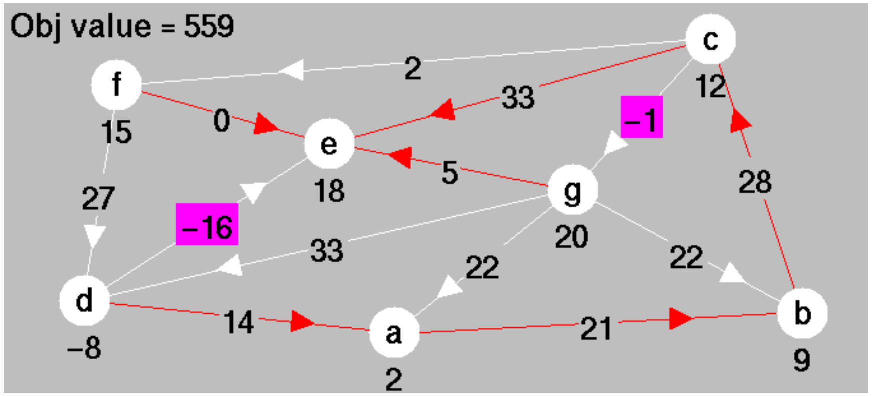
Used when all primal flows are nonnegative (i.e., primal feasible).



Pivot Rules:

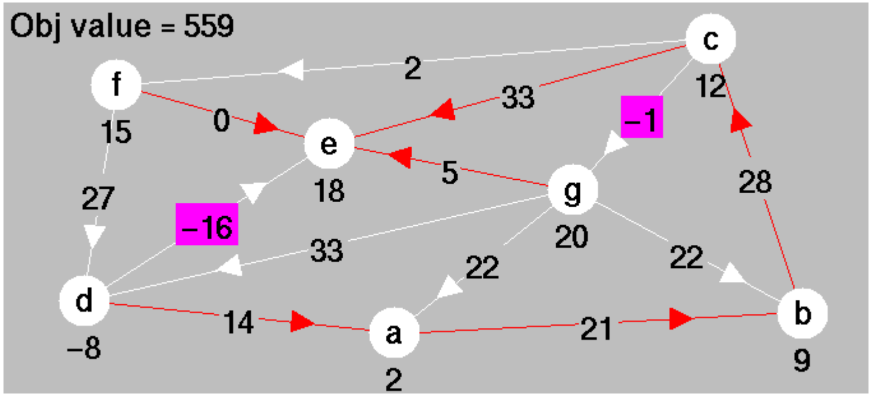
Entering arc: Pick a nontree arc having a negative (i.e. infeasible) dual slack.

Entering arc: (g,e)
Leaving arc: (g,d)



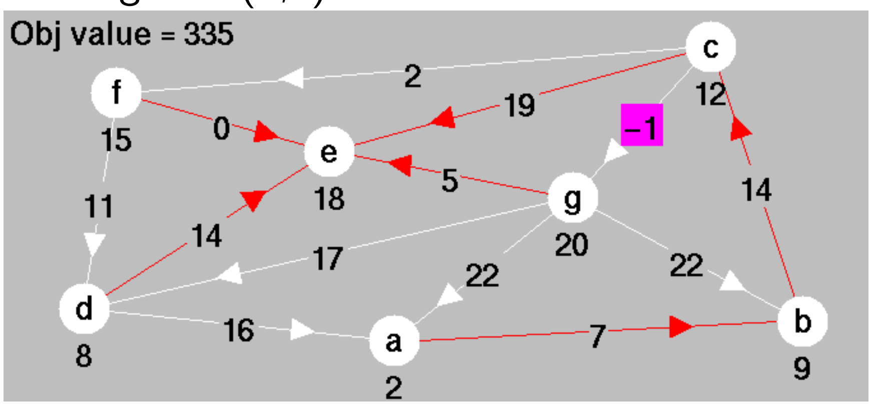
Leaving arc: Add entering arc to make a *cycle*.
Leaving arc is an arc on the cycle, pointing in the *opposite* direction to the entering arc, and
of all such arcs, it is the one with the *smallest* primal flow.

Primal Method—Second Pivot



Entering arc: (d,e)

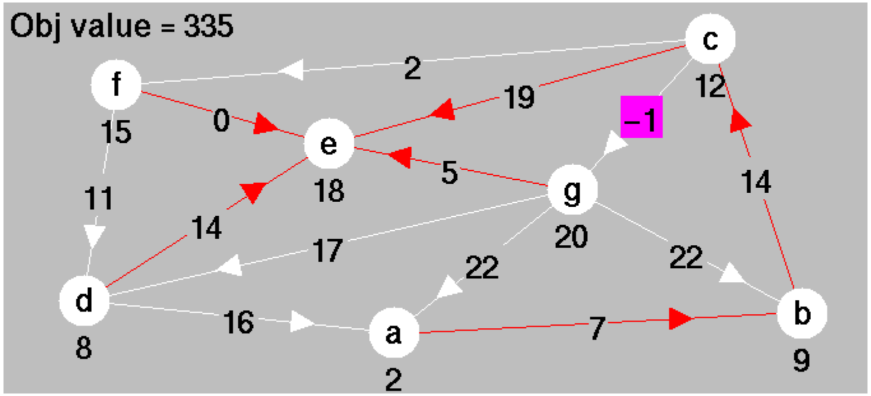
Leaving arc: (d,a)



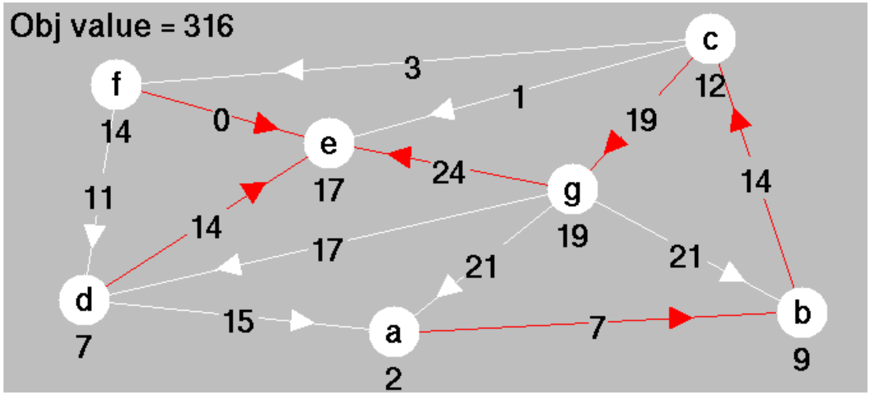
Explanation of leaving arc rule:

- Increase flow on (d,e).
- Each unit increase produces a unit *increase* on arcs pointing in the *same* direction.
- Each unit increase produces a unit *decrease* on arcs pointing in the *opposite* direction.
- The first to reach zero will be the one pointing in the opposite direction and having the smallest flow among all such arcs.

Primal Method—Third Pivot



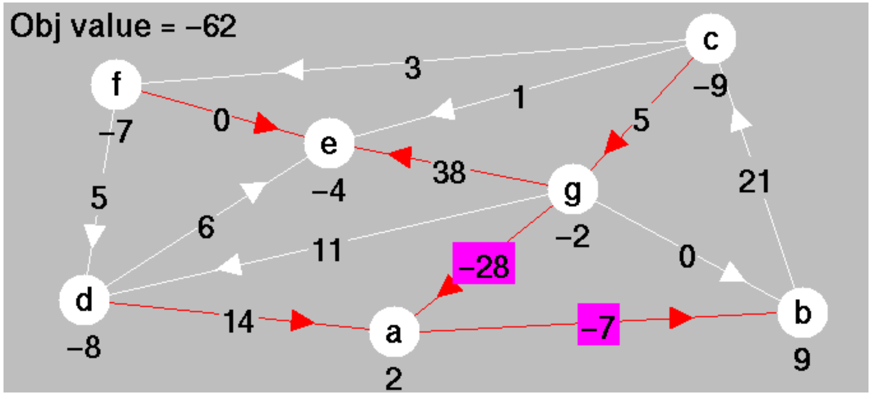
Entering arc: (c,g)
Leaving arc: (c,e)



Optimal!

Dual Network Simplex Method

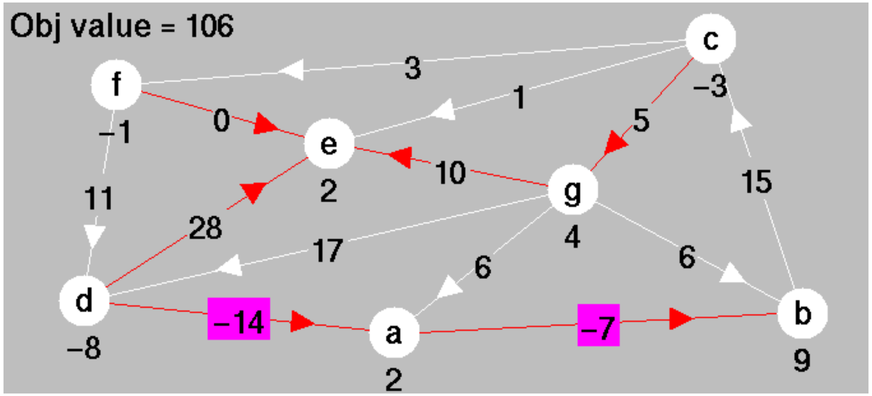
Used when all dual slacks are nonnegative (i.e., dual feasible).



Pivot Rules:

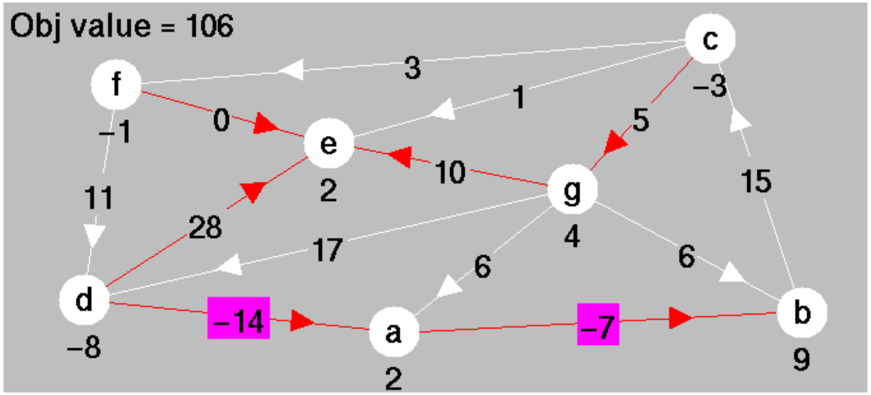
Leaving arc: Pick a tree arc having a negative (i.e. infeasible) primal flow.

Leaving arc: (g,a)
Entering arc: (d,e)

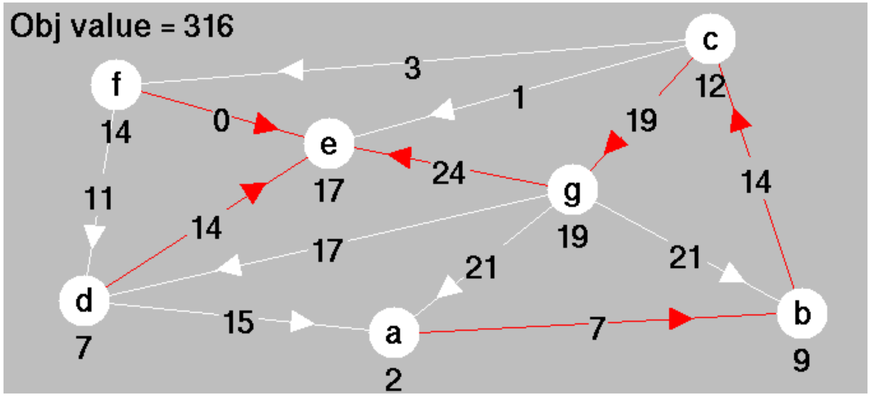


Entering arc: Remove leaving arc to split the spanning tree into two subtrees. Entering arc is an arc reconnecting the spanning tree with an arc in the *opposite* direction, and, of all such arcs, is the one with the *smallest* dual slack.

Dual Network Simplex Method—Second Pivot



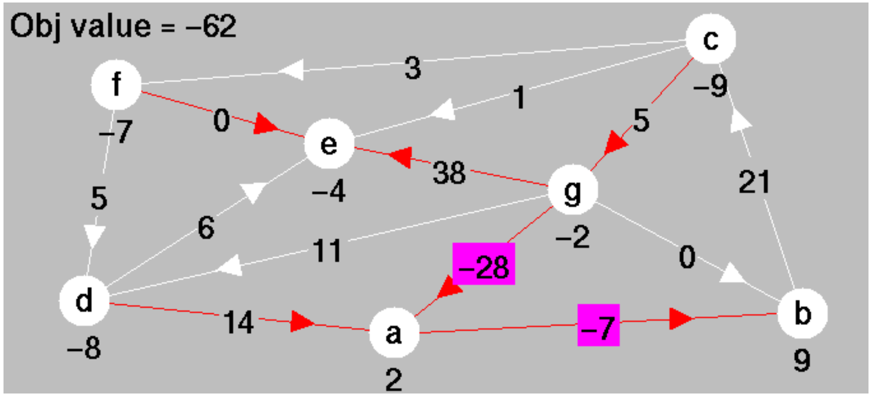
Leaving arc: (d,a)
Entering arc: (b,c)



Optimal!

Explanation of Entering Arc Rule

Recall initial tree solution:

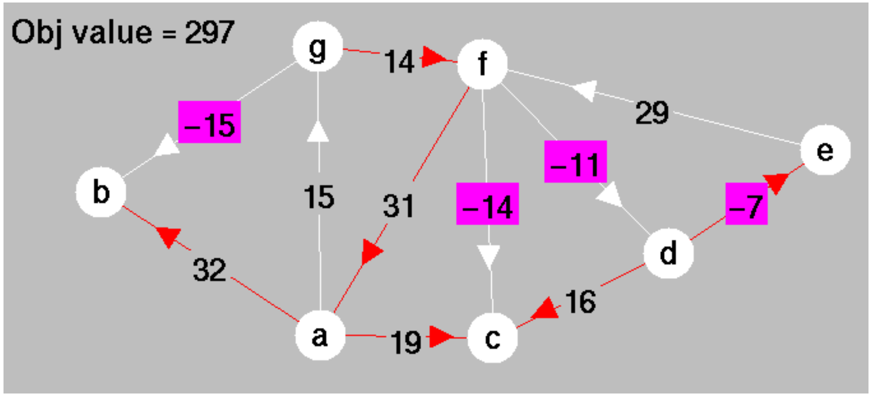


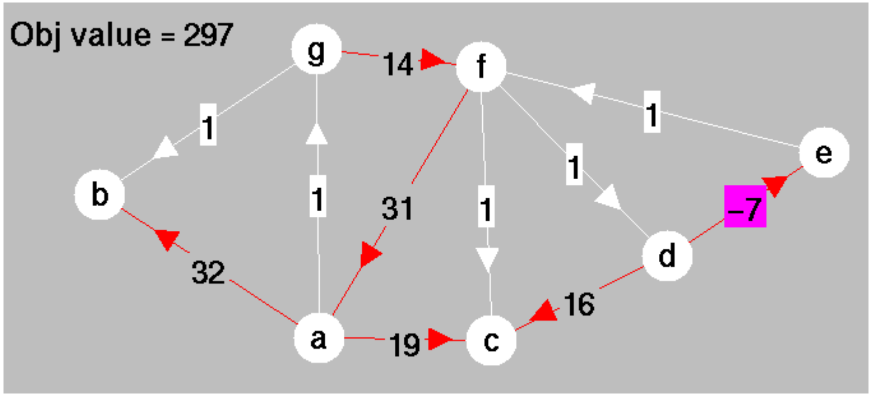
Leaving arc: (g,a)
Entering arc: (d,e)

- Remove leaving arc. Need to find a reconnecting arc.
- Since the leaving arc has a negative flow, there is a net supply at the sub-tree attached to the head node and a net demand at the subtree attached to the tail node.
- So, reconnecting with an arc that spans in the same direction does not improve anything.
- Hence, only consider arcs spanning the two subtrees in the opposite direction.

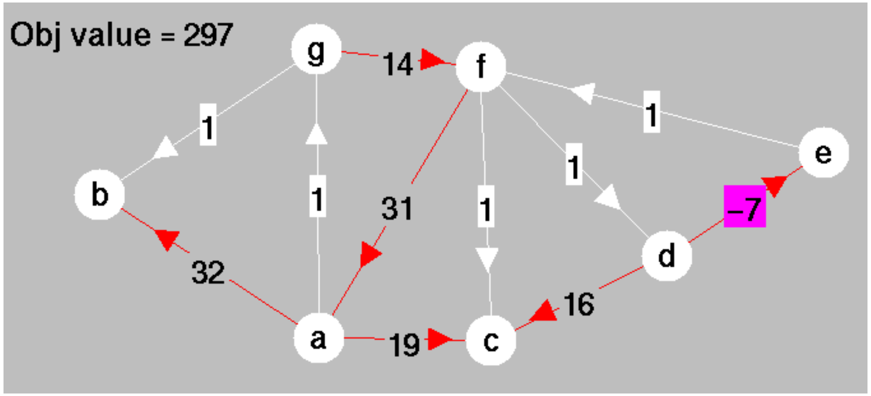
- Consider a potential arc reconnecting in the opposite direction, say (b,c).
  - Its dual slack will drop to zero.
  - All other reconnecting arcs pointing in the same direction will drop by the same amount.
  - To maintain nonnegativity of all the others, must pick the one that drops the least.

Example.

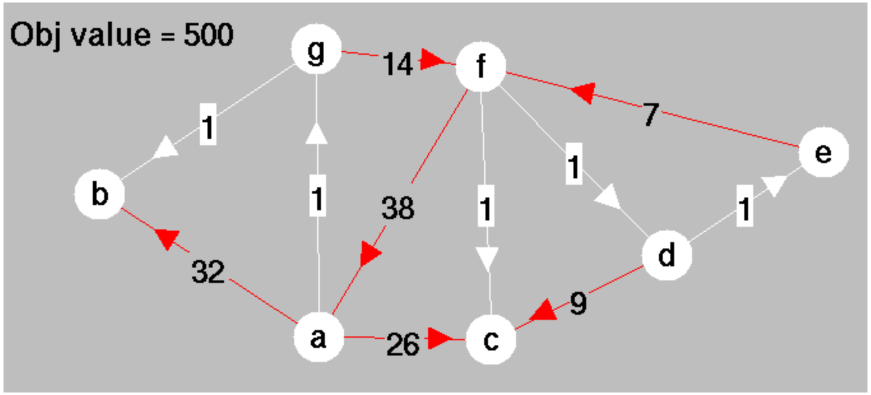




- Turn off display of dual slacks.
- Turn on display of artificial dual slacks.

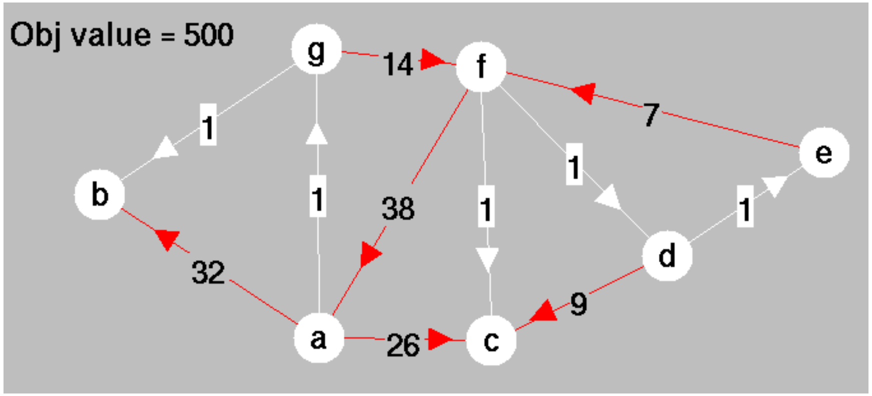


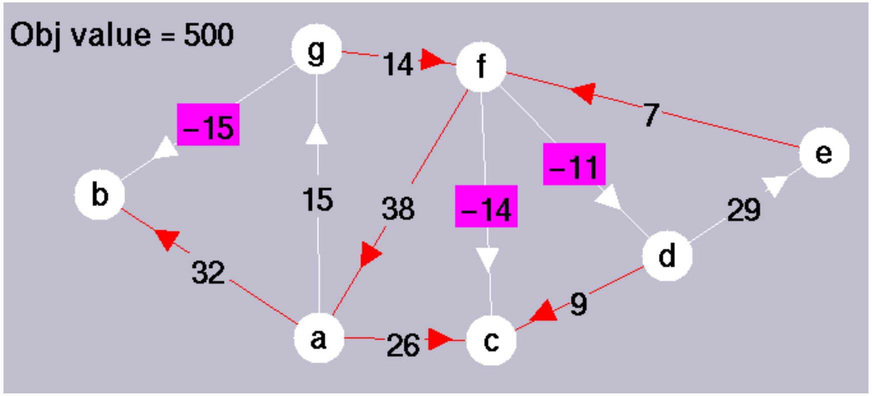
Use dual network simplex method.
Leaving arc: (d,e)   Entering arc: (e,f)



Dual Feasible!

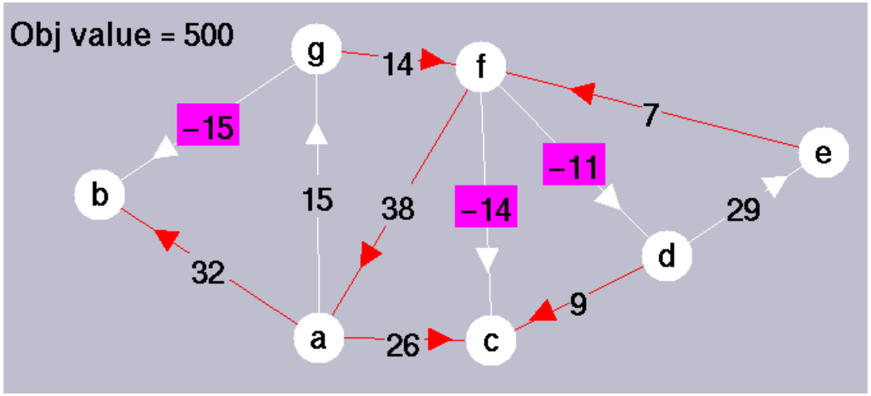
Two-Phase Method–Phase II



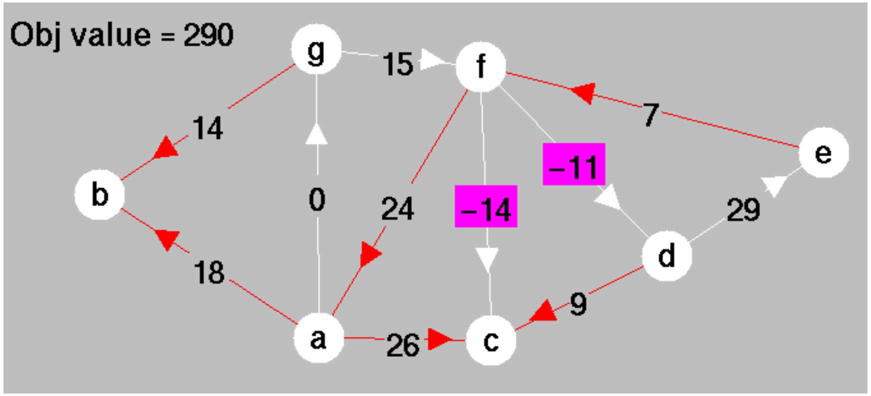


- Turn off display of artificial dual slacks.
- Turn on display of true dual slacks.

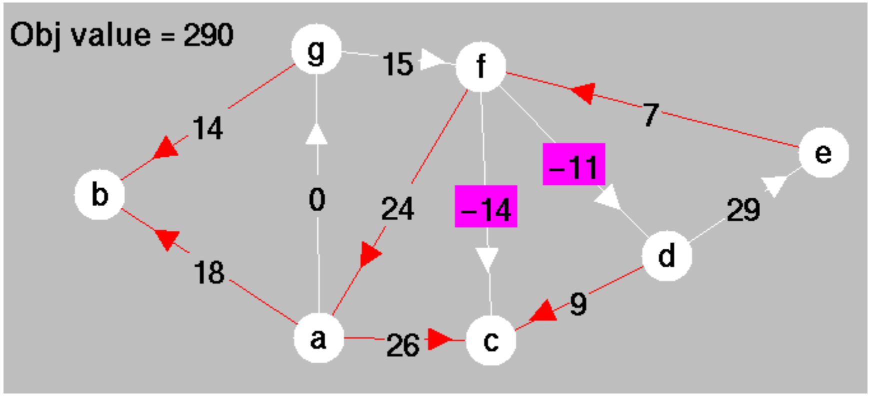
Two-Phase Method–Second Pivot



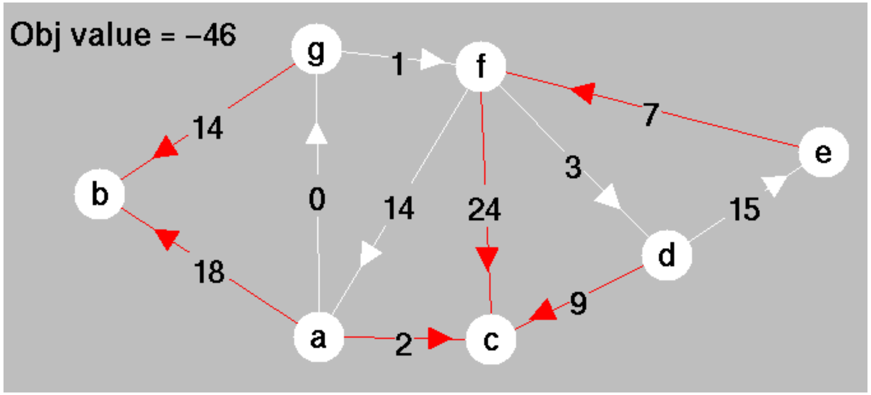
Entering arc: (g,b)
Leaving arc: (g,f)



Two-Phase Method–Third Pivot



Entering arc: (f,c)
Leaving arc: (a,f)



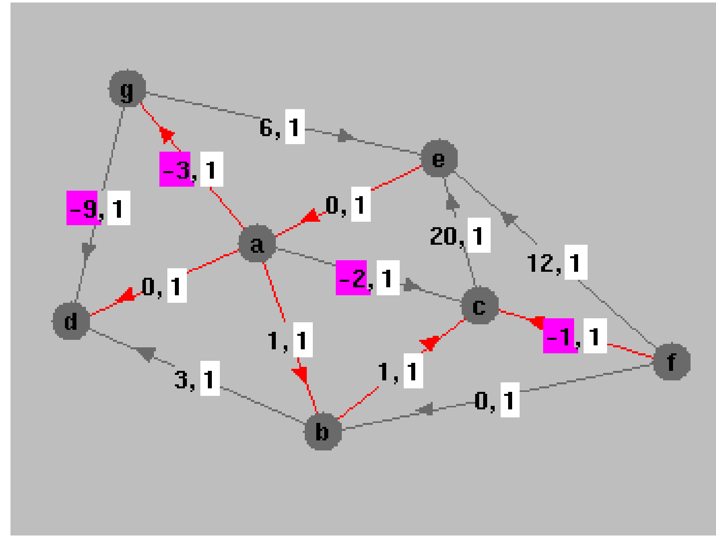
Optimal!

Online Network Simplex Pivot Tool

Click here (or on any displayed network) to try out the online network simplex pivot tool.

Parametric Self-Dual Method

- Artificial flows and slacks are multiplied by a parameter μ.
- In the Figure, $6, 1$ represents $6 + 1\mu$.
- *Question:* For which μ values is dictionary optimal?
- *Answer:*

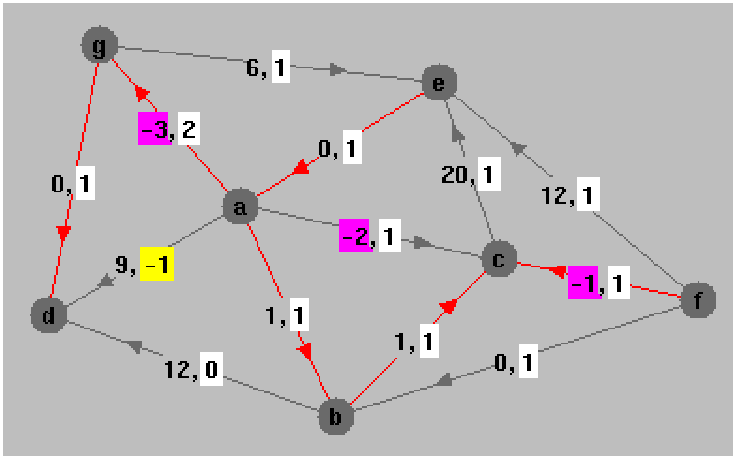


$$\begin{array}{rcrclrrcrcl} 1 & + & \mu & \geq & 0 \quad (a,b) & & & \mu & \geq & 0 \quad (f,b) \\ -2 & + & \mu & \geq & 0 \quad (a,c) & 20 & + & \mu & \geq & 0 \quad (c,e) \\ & & \mu & \geq & 0 \quad (a,d) & -1 & + & \mu & \geq & 0 \quad (f,c) \\ & & \mu & \geq & 0 \quad (e,a) & -9 & + & \mu & \geq & 0 \quad (g,d) \\ -3 & + & \mu & \geq & 0 \quad (a,g) & 12 & + & \mu & \geq & 0 \quad (f,e) \\ & & \mu & \geq & 0 \quad (b,c) & 6 & + & \mu & \geq & 0 \quad (g,e) \\ 3 & + & \mu & \geq & 0 \quad (b,d) & & & & & \end{array} \tag{1}$$

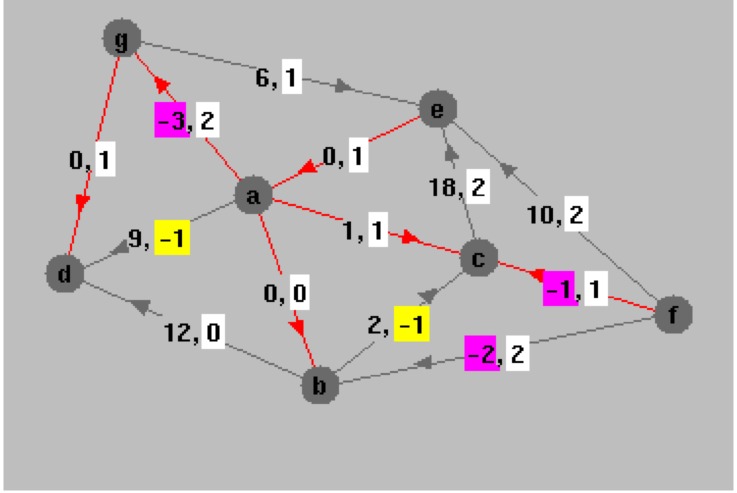
- That is, $9 \leq \mu < \infty$.
- Lower bound on μ is generated by arc (g,d).
- Therefore, (g,d) enters.
- Arc (a,d) leaves.

Second Iteration

- Range of μ values: $2 \leq \mu \leq 9$.
- Entering arc: (a,c)
- Leaving arc: (b,c)

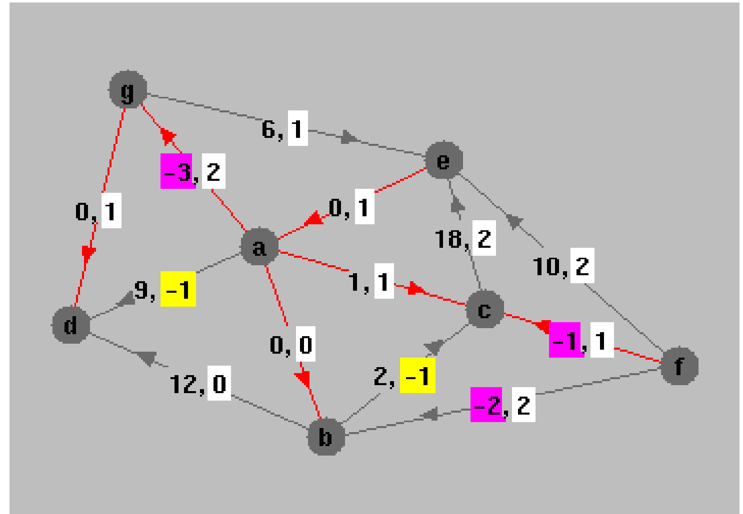


New tree:

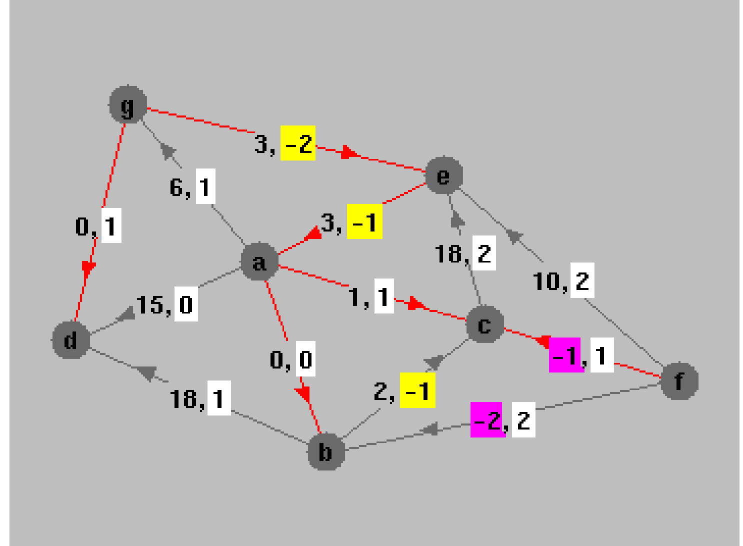


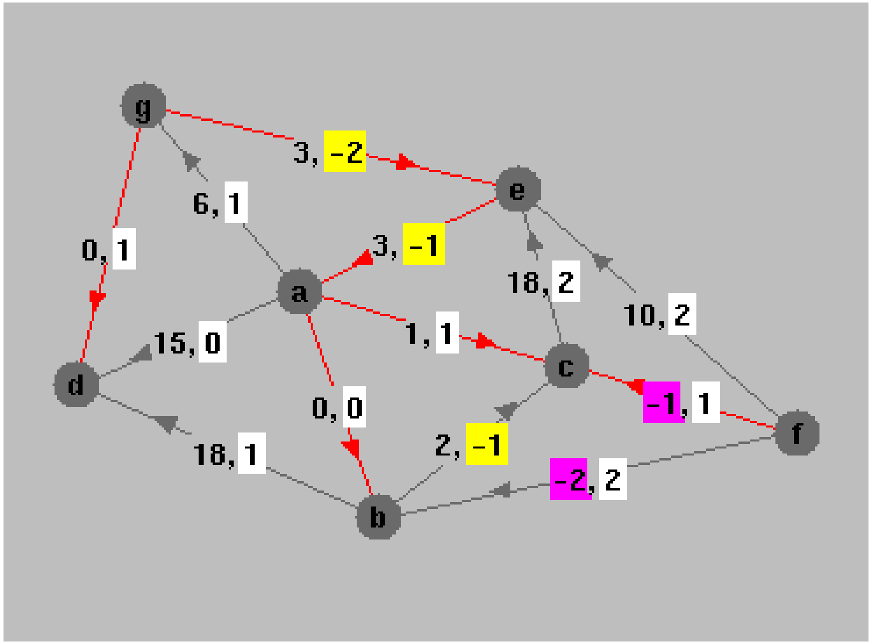
Third Iteration

- Range of μ values:
  $1.5 \leq \mu \leq 2.$
- Leaving arc: (a,g)
- Entering arc: (g,e)



New tree:

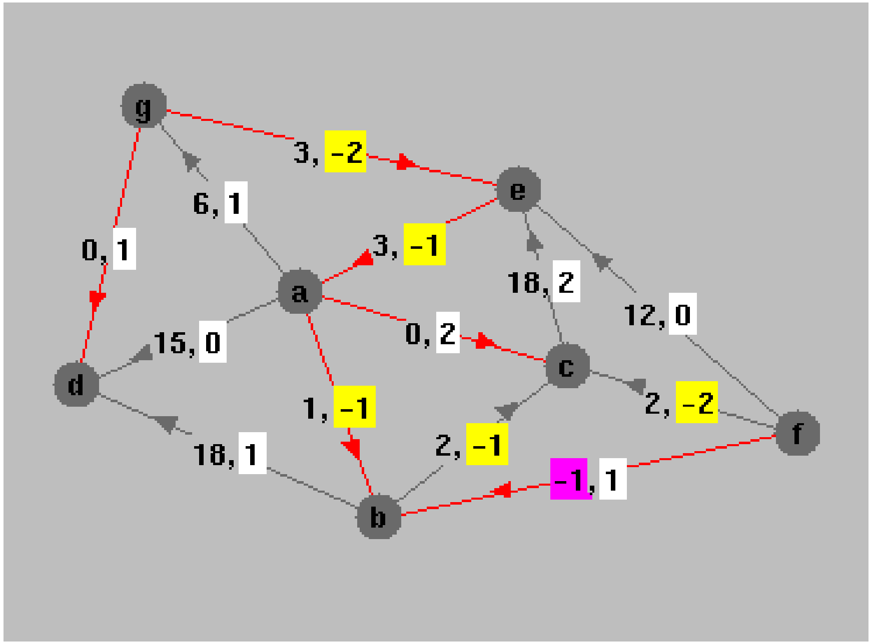




- Range of μ values:
  $1 \leq \mu \leq 1.5$.
- A tie:
  - Arc (f,b) enters, or
  - Arc (f,c) leaves.
- Decide arbitrarily:
  - Leaving arc: (f,c)
  - Entering arc: (f,b)

Fifth Iteration

- Range of μ values: $1 \leq \mu \leq 1$.
- Leaving arc: (f,b)
- Nothing to Enter.



Primal Infeasible!

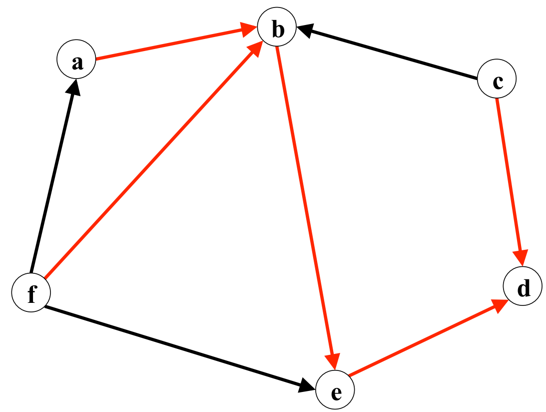
Online Network Simplex Pivot Tool

Click here (or on any displayed network) to try out the online network simplex pivot tool.

Planar Networks

Definition. *Network is called* planar *if can be drawn on a plane without intersecting arcs.*

Theorem. *Every planar network has a* geometric dual*—dual nodes are* faces *of primal network.*



Notes:

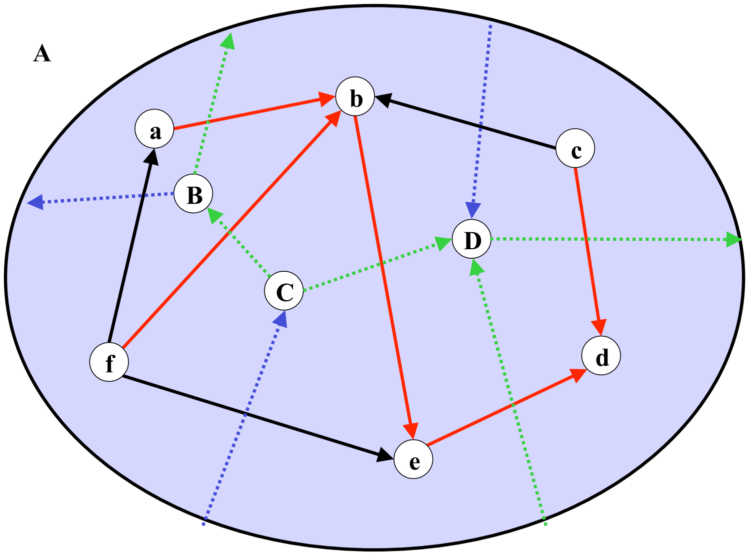
- Dual node A is "node at infinity".
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don't forget node A).

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*

Planar Networks

Definition. *Network is called* planar *if can be drawn on a plane without intersecting arcs.*

Theorem. *Every planar network has a* geometric dual*—dual nodes are* faces *of primal network.*



Notes:

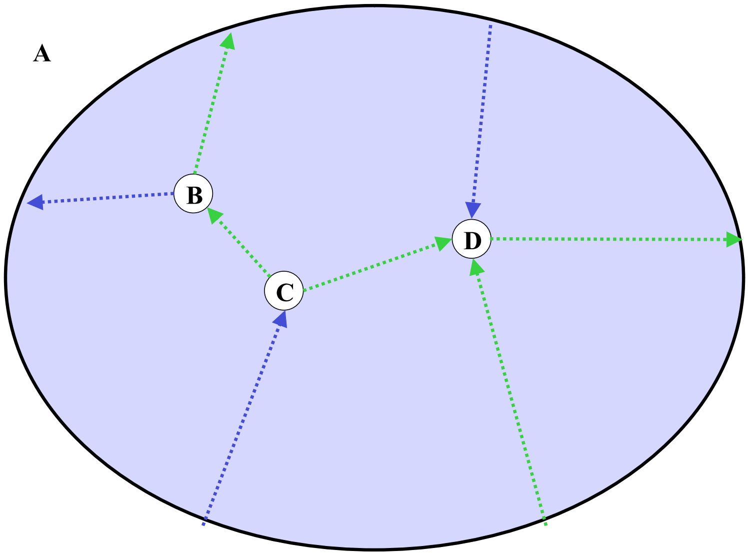
- Dual node A is "node at infinity".
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don't forget node A).

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*

Planar Networks

Definition. *Network is called* planar *if can be drawn on a plane without intersecting arcs.*

Theorem. *Every planar network has a* geometric dual*—dual nodes are* faces *of primal network.*



Notes:

- Dual node A is "node at infinity".
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don't forget node A).

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*

Planar Networks—Algebraic Dual

Primal flow problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = -b \\ & x \geq 0 \end{array}$$

There is one redundant equation. Drop it and rewrite the equations:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_0 x = -b_0 \\ & x \geq 0 \end{array}$$

A spanning tree corresponds to a basic solution:

$$A_0 = [B \ N].$$

The dual has free variables and slacks:

$$\begin{array}{ll} \text{maximize} & -\begin{bmatrix} b_0 \\ 0 \end{bmatrix}^T \begin{bmatrix} y \\ z \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} B^T & I & 0 \\ N^T & 0 & I \end{bmatrix} \begin{bmatrix} y \\ z_{\mathcal{B}} \\ z_{\mathcal{N}} \end{bmatrix} = \begin{bmatrix} c_{\mathcal{B}} \\ c_{\mathcal{N}} \end{bmatrix} \\ & z \geq 0 \end{array}$$

Solve for y:

$$y = B^{-T}(c_{\mathcal{B}} - z_{\mathcal{B}}).$$

Eliminate from problem:

$$\begin{array}{ll} \text{maximize} & -(B^{-1}b_0)^T(c_{\mathcal{B}} - z_{\mathcal{B}}) \\ \text{subject to} & \begin{bmatrix} -(B^{-1}N)^T & I \end{bmatrix} \begin{bmatrix} z_{\mathcal{B}} \\ z_{\mathcal{N}} \end{bmatrix} \\ & \quad = c_{\mathcal{N}} - (B^{-1}N)^T c_{\mathcal{B}} \\ & z \geq 0 \end{array}$$

Matlab Check

```
% a node/arc incidence matrix
A = [-1  0  0  0  0  1  0  0;...
      1 -1  1  0  0  0  1  0;...
      0  0 -1 -1  0  0  0  0;...
      0  0  0  1  1  0  0  0;...
      0  1  0  0 -1  0  0  1;...
      0  0  0  0  0 -1 -1 -1];
[m,n] = size(A);
A0 = A(1:end-1,:);   % drop last row
B=A0(:,[1 2 4 5 7]);  % these are the tree arcs
N=A0(:,[3 6 8]); % these are the nontree arcs
Binv = inv(B);
BinvN = Binv*N;
BN = [B N];
alg_dual = [-BinvN' eye(n-m+1)]
BinvNt = BinvN'

% node/arc incidence matrix for graphical dual
AA = [ 1  0 -1  1 -1  1  0 -1;...
      -1  0  0  0  0 -1  1  0;...
       0 -1  0  0  0  0 -1  1;...
       0  1  1 -1  1  0  0  0];
AA0 = AA(1:end-1,:); % drop last row
geom_dual = AA
BB = AA0(:,[3 6 8]);
NN = AA0(:,[1 2 4 5 7]);
BBinvNN = inv(BB)*NN
```

Matlab Output

```
alg_dual =
     0     1    -1     1     0     1     0     0
     1     0     0     0    -1     0     1     0
     0    -1     0     0    -1     0     0     1

BinvNt =
     0    -1     1    -1     0
    -1     0     0     0     1
     0     1     0     0     1

geom_dual =
     1     0    -1     1    -1     1     0    -1
    -1     0     0     0     0    -1     1     0
     0    -1     0     0     0     0    -1     1
     0     1     1    -1     1     0     0     0

BBinvNN =
     0     1    -1     1     0
     1     0     0     0    -1
     0    -1     0     0    -1
```

Notes:

- If the middle row of `alg_dual` is negated, then the matrix can be made into a node/arc incidence matrix by adding an appropriately chosen forth row. In general it is not obvious how to connect `alg_dual` to `geom_dual`.
- But, `BinvNt` = -`BBinvNN`.

Euler's Formula

Dimension of primal problem with (one) redundant equation:

$$m \times n$$

Dimension of primal problem without redundant equation:

$$(m - 1) \times n$$

Dimension of dual problem with slacks, free variables, but without redundancy:

$$n \times (m - 1 + n)$$

Dimension of dual problem without free variables and without redundancy:

$$(n - (m - 1)) \times n$$

Number of faces equals number of dual nodes:

$$n - m + 2$$

Euler's formula:

$$\text{nodes} - \text{arcs} + \text{faces} = m - n + (n - m + 2) = 2$$

Integrality Theorem

Theorem. *Assuming integer data, every basic feasible solution assigns integer flow to every arc.*

Corollary. *Assuming integer data, every basic optimal solution assigns integer flow to every arc.*