Linear Programming: Chapter 1
Introduction

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Resource Allocation

$$
\begin{array}{lrcrcrcl}
\text{maximize} & c_1 x_1 & + & c_2 x_2 & + \cdots + & c_n x_n & & \\
\text{subject to} & a_{11} x_1 & + & a_{12} x_2 & + \cdots + & a_{1n} x_n & \leq & b_1 \\
& a_{21} x_1 & + & a_{22} x_2 & + \cdots + & a_{2n} x_n & \leq & b_2 \\
& & & & \vdots & & & \\
& a_{m1} x_1 & + & a_{m2} x_2 & + \cdots + & a_{mn} x_n & \leq & b_m \\
& & & & x_1, \; x_2, \ldots, & x_n & \geq & 0,
\end{array}
$$

where

$$
\begin{aligned}
c_j &= \text{profit per unit of product } j \text{ produced} \\
b_i &= \text{units of raw material } i \text{ on hand} \\
a_{ij} &= \text{units of raw material } i \text{ required to produce one unit of product } j.
\end{aligned}
$$

Blending Problems (Diet Problem)

$$
\begin{array}{lrcrcrcrcl}
\text{minimize} & & & c_1x_1 & + & c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{subject to} & l_1 & \le & a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & \le & u_1 \\
& l_2 & \le & a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & \le & u_2 \\
& & & & & & \vdots & & & \\
& l_m & \le & a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \le & u_m \\
& & & x_1, & & x_2, \ldots, \; x_n & \ge & 0 & , &
\end{array}
$$

where

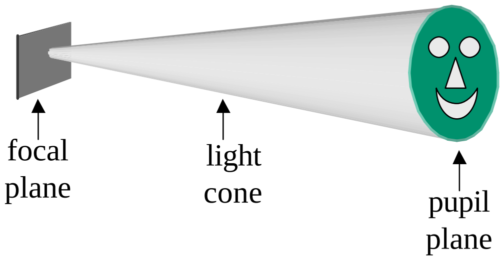
$$
\begin{aligned}
c_j &= \text{cost per unit of food } j \\
l_i &= \text{minimum daily allowance of nutrient } i \\
u_i &= \text{maximum daily allowance of nutrient } i \\
a_{ij} &= \text{units of nutrient } i \text{ contained in one unit of food } j.
\end{aligned}
$$

Shape Optimization (Telescope Design)

The problem is to design and build a space telescope that will be able to "see" planets around nearby stars (other than the Sun).

Consider a telescope. Light enters the front of the telescope. This is called the *pupil plane*.

The telescope focuses all the light passing through the pupil plane from a given direction at a certain point on the *focal plane*, say $(0, 0)$.



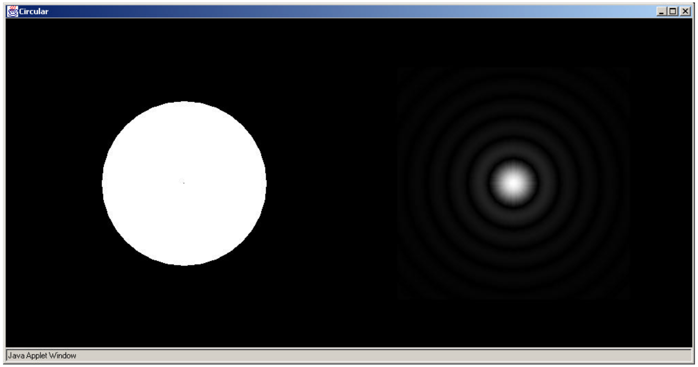
However, the wave nature of light makes it impossible to concentrate all of the light at a point. Instead, a small disk, called the *Airy disk*, with diffraction rings around it appears.

These diffraction rings are bright relative to any planet that might be orbiting a nearby star and so would completely hide the planet. The Sun, for example, would appear 10^{10} times brighter than the Earth to a distant observer.

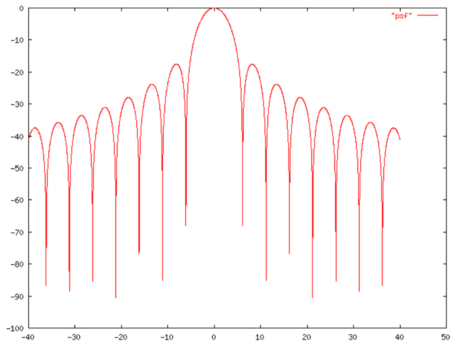
By placing a mask over the pupil, one can design the shape and strength of the diffraction rings. The problem is to find an optimal shape so as to put a very deep *null* very close to the Airy disk.

Airy Disk and Diffraction Rings

A conventional telescope has a circular openning as depicted by the left side of the figure. Visually, a star then looks like a small disk with rings around it, as depicted on the right.

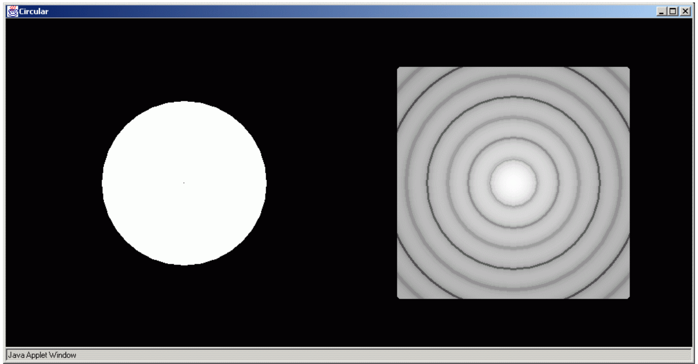


The rings grow progressively dimmer as this log-plot shows:



Airy Disk and Diffraction Rings—Log Scaling

Here's the same Airy disk from the previous slide plotted using a logarithmic brightness scale with $10^{-11} = -110$dB set to black:



The problem is to find an aperture mask, i.e. a pupil plane mask, that yields a -100 dB *null* somewhere near the first diffraction ring. *A hard problem!* Such a null would appear almost black in this log-scaled image.

Electric Field

Consider “tinting” the front opening of the telescope using a nonuniform tint given by $A(r)$.

In such a situation, the image plane electric field of a star is a rotationally symmetric real function $E(\rho)$:

$$E(\rho) = 2\pi \int_0^{D/2} A(r) J_0(2\pi r \rho) r dr$$

The intensity of the light at radius ρ from the center of the image plane is given by the square of the electric field.

Maximizing Throughput

We maximize the "area" under $A(r)$ (make the tinting as bright as possible) subject to very strict contrast constraints:

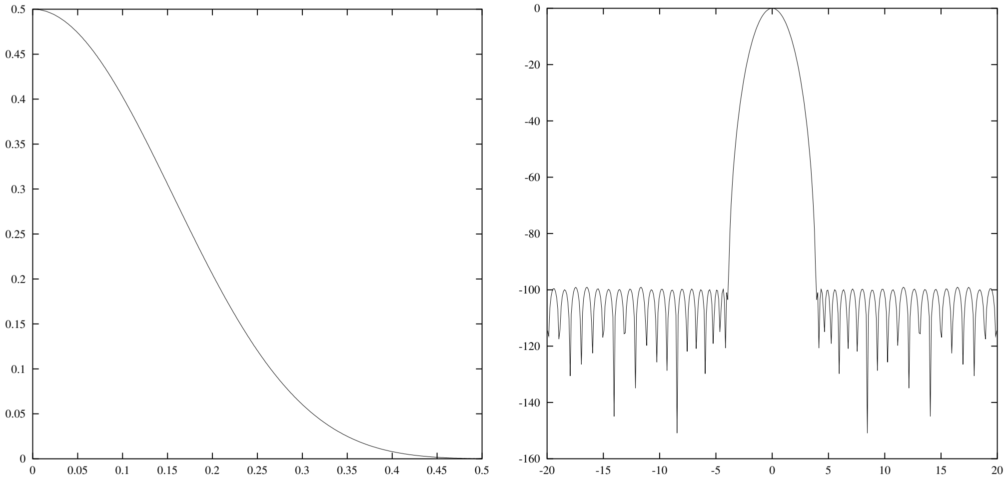
$$\text{maximize} \int_0^{D/2} A(r) r dr$$

$$\begin{aligned} \text{subject to } & -10^{-5}E(0) \le E(\rho) \le 10^{-5}E(0), && \text{for } \rho_{\min} \le \rho \le \rho_{\max} \\ & 0 \le A(r) \le 1, && \text{for } 0 \le r \le D/2 \end{aligned}$$

The first constraint guarantees 10^{-10} light intensity throughout a desired annulus of the focal plane, and the remaining constraint ensures that the tinting is really a tinting.

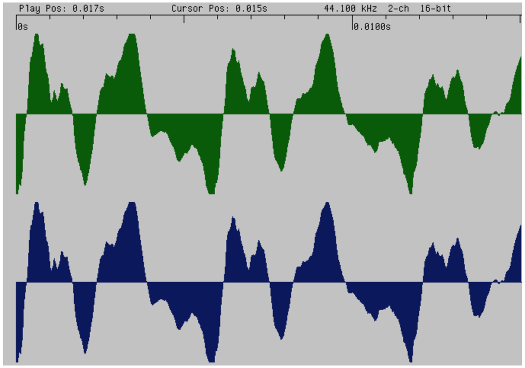
Solution via Linear Programming

$\mathcal{O} = \{(\xi, 0) : \xi_0 \leq \xi \leq \xi_1\}$
$\rho_{\min} = 4$
$\rho_{\max} = 60$
Thruput $= 10\%$



Finite Impulse Response (FIR) Filter Design

- Audio is stored digitally in a computer as a stream of short integers: u_k, $k \in \mathbb{Z}$.
- When the music is played, these integers are used to drive the displacement of the speaker from its resting position.
- For CD quality sound, 44100 short integers get played per second per channel.



0	-32768	8	-23681	16	12111
1	-32768	9	-18449	17	17311
2	-32768	10	-11025	18	21311
3	-30753	11	-6913	19	23055
4	-28865	12	-4337	20	23519
5	-29105	13	-1329	21	25247
6	-29201	14	1743	22	27535
7	-26513	15	6223	23	29471

FIR Filter Design—Continued

- A *finite impulse response (FIR) filter* takes as input a digital signal and convolves this signal with a finite set of fixed numbers $h_0, \ldots, h_n$ to produce a filtered output signal:

$$y_k = \sum_{i=-n}^{n} h_{|i|} u_{k-i}.$$

- Sparing the details, the output power at frequency ν is given by

$$|H(\nu)|^2$$

where

$$H(\nu) = \sum_{k=-n}^{n} h_{|k|} e^{2\pi i k \nu} = h(0) + 2 \sum_{k=1}^{n} h_k \cos(2\pi k \nu),$$

- Similarly, the mean absolute deviation from a flat frequency response over a frequency range, say $\mathcal{L} \subset [0, 1]$, is given by

$$\frac{1}{|\mathcal{L}|} \int_{\mathcal{L}} |H(\nu) - 1| \, d\nu$$

Filter Design: Woofer, Midrange, Tweeter

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\int_0^1 |H_w(\nu) + H_m(\nu) + H_t(\nu) - 1| \, d\nu \\
\text{subject to} & -\epsilon \le H_w(\nu) \le \epsilon, \qquad \nu \in W = [.2, .8] \\
& -\epsilon \le H_m(\nu) \le \epsilon, \qquad \nu \in M = [.4, .6] \cup [.9, .1] \\
& -\epsilon \le H_t(\nu) \le \epsilon, \qquad \nu \in T = [.7, .3]
\end{array}
$$

where

$$
\begin{aligned}
H_i(\nu) &= h_0^i + 2 \sum_{k=1}^{n} h_k^i \cos(2\pi k \nu), \quad i = W, M, T \\
h_k^i &= \text{filter coefficients, i.e., } \textbf{decision variables}
\end{aligned}
$$

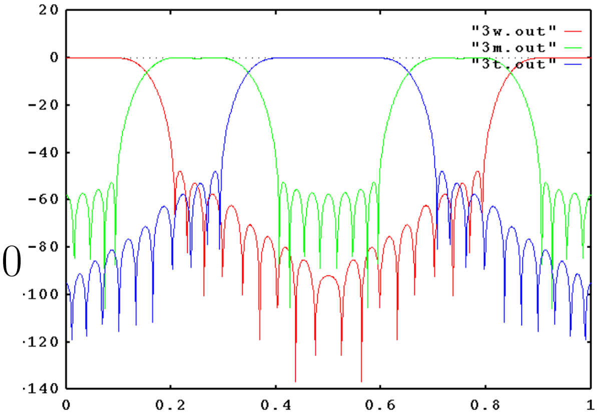
Conversion to a Linear Programming Problem

$$\begin{aligned}
\text{minimize} \quad & \int_0^1 t(\nu)d\nu \\
\text{subject to} \quad & t(\nu) \leq H_w(\nu) + H_m(\nu) + H_t(\nu) - 1 \leq t(\nu) \quad && \nu \in [0,1] \\
& -\epsilon \leq H_w(\nu) \leq \epsilon, && \nu \in W \\
& -\epsilon \leq H_m(\nu) \leq \epsilon, && \nu \in M \\
& -\epsilon \leq H_t(\nu) \leq \epsilon, && \nu \in T
\end{aligned}$$

Specific Example

filter length: $n = 14$

integral discretization: $N = 1000$



Demo: orig-clip woofer midrange tweeter reassembled

Ref: J.O. Coleman, U.S. Naval Research Laboratory,

CISS98 paper available: engr.umbc.edu/∼jeffc/pubs/abstracts/ciss98.html

Portfolio Optimization

Markowitz Shares the 1990 Nobel Prize

Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR'S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize in Economic Sciences with one third each, to

Professor **Harry Markowitz**, City University of New York, USA,
Professor **Merton Miller**, University of Chicago, USA,
Professor **William Sharpe**, Stanford University, USA,

for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called, *Capital Asset Pricing Model* (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary
Financial markets serve a key purpose in a modern market economy by allocating productive resources among various areas of production. It is to a large extent through financial markets that saving in different sectors of the economy is transferred to firms for investments in buildings and machines. Financial markets also reflect firms' expected prospects and risks, which implies that risks can be spread and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry Markowitz who developed a theory for households' and firms' allocation of financial assets under uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally invested in assets which differ in regard to their expected return and risk, and thereby also how risks can be reduced.

Historical Data

Year	US 3-Month T-Bills	US Gov. Long Bonds	S&P 500	Wilshire 5000	NASDAQ Composite	Lehman Bros. Corp. Bonds	EAFE	Gold
1973	1.075	0.942	0.852	0.815	0.698	1.023	0.851	1.677
1974	1.084	1.020	0.735	0.716	0.662	1.002	0.768	1.722
1975	1.061	1.056	1.371	1.385	1.318	1.123	1.354	0.760
1976	1.052	1.175	1.236	1.266	1.280	1.156	1.025	0.960
1977	1.055	1.002	0.926	0.974	1.093	1.030	1.181	1.200
1978	1.077	0.982	1.064	1.093	1.146	1.012	1.326	1.295
1979	1.109	0.978	1.184	1.256	1.307	1.023	1.048	2.212
1980	1.127	0.947	1.323	1.337	1.367	1.031	1.226	1.296
1981	1.156	1.003	0.949	0.963	0.990	1.073	0.977	0.688
1982	1.117	1.465	1.215	1.187	1.213	1.311	0.981	1.084
1983	1.092	0.985	1.224	1.235	1.217	1.080	1.237	0.872
1984	1.103	1.159	1.061	1.030	0.903	1.150	1.074	0.825
1985	1.080	1.366	1.316	1.326	1.333	1.213	1.562	1.006
1986	1.063	1.309	1.186	1.161	1.086	1.156	1.694	1.216
1987	1.061	0.925	1.052	1.023	0.959	1.023	1.246	1.244
1988	1.071	1.086	1.165	1.179	1.165	1.076	1.283	0.861
1989	1.087	1.212	1.316	1.292	1.204	1.142	1.105	0.977
1990	1.080	1.054	0.968	0.938	0.830	1.083	0.766	0.922
1991	1.057	1.193	1.304	1.342	1.594	1.161	1.121	0.958
1992	1.036	1.079	1.076	1.090	1.174	1.076	0.878	0.926
1993	1.031	1.217	1.100	1.113	1.162	1.110	1.326	1.146
1994	1.045	0.889	1.012	0.999	0.968	0.965	1.078	0.990

Notation: $R_j(t)$ = return on investment j in time period t.

Risk vs. Reward

Reward—estimated using historical means:

$$\text{reward}_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t).$$

Risk—Markowitz defined risk as the variability of the returns as measured by the historical variances:

$$\text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} \left(R_j(t) - \text{reward}_j \right)^2 .$$

However, to get a linear programming problem (and for other reasons) we use the sum of the absolute values instead of the sum of the squares:

$$\text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} \left| R_j(t) - \text{reward}_j \right| .$$

Hedging

Investment A: up 20%, down 10%, equally likely—a risky asset.

Investment B: up 20%, down 10%, equally likely—another risky asset.

Correlation: up years for A are down years for B and vice versa.

Portfolio—half in A, half in B: up 5% every year! No risk!

Portfolios

Fractions: x_j = fraction of portfolio to invest in j.

Portfolio's Historical Returns:

$$R(t) = \sum_j x_j R_j(t)$$

Portfolio's Reward:

$$\text{reward}(x) = \frac{1}{T} \sum_{t=1}^{T} R(t) = \frac{1}{T} \sum_{t=1}^{T} \sum_j x_j R_j(t)$$

Portfolio's Risk:

$$\begin{aligned}
\mathsf{risk}(x) &= \frac{1}{T}\sum_{t=1}^{T}\left|R(t) - \mathsf{reward}(x)\right| \\
&= \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} x_j R_j(t) - \frac{1}{T}\sum_{s=1}^{T}\sum_{j} x_j R_j(s)\right| \\
&= \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} x_j \left(R_j(t) - \frac{1}{T}\sum_{s=1}^{T} R_j(s)\right)\right| \\
&= \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} x_j (R_j(t) - \mathsf{reward}_j)\right|
\end{aligned}$$

A Markowitz-Type Model

Decision Variables: the fractions x_j.

Objective: maximize return, minimize risk.

Fundamental Lesson: can't simultaneously optimize two objectives.

Compromise: set an upper bound μ for risk and maximize reward subject to this bound constraint:

- Parameter μ is called risk aversion parameter.
- Large value for μ puts emphasis on reward maximization.
- Small value for μ puts emphasis on risk minimization.

Constraints:

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\left|\sum_j x_j(R_j(t) - \text{reward}_j)\right| &\leq \mu \\
\sum_j x_j &= 1 \\
x_j &\geq 0 \qquad \text{for all } j
\end{aligned}$$

Optimization Problem

$$\begin{aligned}
\text{maximize} \quad & \frac{1}{T}\sum_{t=1}^{T}\sum_{j} x_j R_j(t) \\
\text{subject to} \quad & \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} x_j (R_j(t) - \text{reward}_j)\right| \le \mu \\
& \sum_{j} x_j = 1 \\
& x_j \ge 0 \qquad \text{for all } j
\end{aligned}$$

Because of absolute values not a linear programming problem.

Easy to convert...

A Linear Programming Formulation

$$
\begin{aligned}
\text{maximize} \quad & \frac{1}{T}\sum_{t=1}^{T}\sum_{j} x_j R_j(t) \\
\text{subject to} \quad & -y_t \le \sum_j x_j (R_j(t) - \mathsf{reward}_j) \le y_t \qquad \text{for all } t \\
& \frac{1}{T}\sum_{t=1}^{T} y_t \le \mu \\
& \sum_j x_j = 1 \\
& x_j \ge 0 \qquad \text{for all } j
\end{aligned}
$$

Efficient Frontier

Varying risk bound μ produces the so-called *efficient frontier*.
Portfolios on the efficient frontier are reasonable.
Portfolios not on the efficient frontier can be strictly improved.

μ	US 3-Month T-Bills	Lehman Bros. Corp. Bonds	NASDAQ Comp.	Wilshire 5000	Gold	EAFE	Reward	Risk
0.1800					0.017	0.983	1.141	0.180
0.1538					0.191	0.809	1.139	0.154
0.1275				0.119	0.321	0.560	1.135	0.128
0.1013				0.407	0.355	0.238	1.130	0.101
0.0751			0.340	0.180	0.260	0.220	1.118	0.075
0.0488	0.172	0.492			0.144	0.008	1.104	0.049
0.0226	0.815	0.100	0.037		0.041	0.008	1.084	0.022

Efficient Frontier