ORF 522
Linear Optimization

Lecture 21

More Applications

Options Pricing

Farkas' Lemma for Systems in Equality Form

Recall Farkas' Lemma:

Lemma. The system $Ax \leq b$ has no solutions if and only if there is a y such that

$$\begin{aligned} A^T y &= 0 \\ y &\geq 0 \\ b^T y &< 0. \end{aligned}$$

Today we need it in another form:

Lemma. The system $Ax = b$, $x \geq 0$ has no solutions if and only if there is a y such that

$$\begin{aligned} A^T y &\geq 0 \\ b^T y &< 0. \end{aligned}$$

Proof is completely analogous to the one we had before. Hence, omitted.

Asset Pricing

Consider a collection of n assets (possible investments).

Suppose that one time period will result in one of m possible scenarios of outcomes.

Let

$$r_{ij} = \text{ return from asset } j \text{ under scenario } i$$

and

$$R = [r_{ij}] .$$

Note: these returns are in dollars-returned per item-of-investment as opposed to our Markowitz model in which returns were measured in dollars-returned per dollar-invested.

Problem: Determine a consistent set of prices for the investments:

$$p_j = \text{ price (in dollars) for asset } j .$$

Arbitrage

Big Assumption 1: We can hold positive or negative quantities of each asset—the return is the same.

Never satisfied in practice. If I give a bank 1 dollar to hold, they will return it after a year with 4% interest but if I give a bank -1 dollar to hold (i.e., I borrow a dollar), they will give -1 back to me with 10% interest.

It is, however, generally assumed to be true, at least for the big players.

Let

$$x_j = \text{ number of units of asset } j \text{ I hold.}$$

Wealth at end of time period under scenario i:

$$w_i = \sum_j r_{ij} x_j.$$

In matrix notation:

$$w = Rx.$$

Recall: the total current "price" for this portfolio is:

$$p^T x$$

An arbitrage is a portfolio x which is guaranteed (under every scenario) to have nonnegative value at the end of the time period but which has a negative price at the beginning:

$$Rx \geq 0 \qquad \text{and} \qquad p^T x < 0.$$

Big Assumption 2: The scenarios considered cover all possibilities.

Efficient Market Assumption

Assumption: Prices will equilibrate so as to eliminate arbitrage.

Theorem. There is no arbitrage if and only if there is a vector y that satisfies:

$$\begin{aligned} R^T y &= p \\ y &\geq 0. \end{aligned}$$

Proof. Immediate from Farkas' Lemma ($A = R^T$, $b = p$, and x and y interchanged).

Notes.

- If $m = n$ and R is nonsingular, then the equality constraints uniquely determine y. Then, only need to check nonnegativity.
- If p is arbitrage-free, then any nonnegative constant times p is too. Therefore, at least one of the prices in p needs to be fixed arbitrarily (by, e.g., Alan Greenspan).

Options

Definition. An option is a contract giving one the "option" to buy a specific stock at a specific price at a specific time in the future.

The price, usually denote K, is called the strike price.

Consider a single-time-period market consisting of

- A Stock
- A Bond
- An Option on the Stock

Let $\bar{S}$ denote the value of the stock at the end of the time period.

If $\bar{S} > K$, then the option holder will exercise the option by buying the stock at K dollars and immediately selling it for $\bar{S}$ dollars, yielding a profit of $\bar{S} - K$ dollars.

If $\bar{S} \leq K$, then the option holder will let the option expire and so at the end the value to the holder is zero dollars.

To summarize: the value at the end of the time period is

$$\max(0, \bar{S} - K).$$

The fundamental question is: how much should one pay for such an option?

Options Pricing

Suppose that there are only two scenarios:

- The stock goes up by a factor $u > 1$, or
- down by a factor $d < 1$.

Under both scenarios, the bond goes up by a factor $r > 1$.

Suppose at the beginning that the stock price is S, the price of the bond is B, and of course the price of the option is to be determined. Let's denote it by O.

The matrix R is then given by:

$$R = \begin{bmatrix} Su & Sd \\ Br & Br \\ \max(0, Su - K) & \max(0, Sd - K) \end{bmatrix}$$

and the vector p is given by:

$$p = \begin{bmatrix} S \\ B \\ O \end{bmatrix}$$

The no-arbitrage theorem says there must exist a vector $y = \begin{bmatrix} y_u & y_d \end{bmatrix}^T$ such that

$$\begin{bmatrix} Su & Sd \\ Br & Br \\ \max(0, Su - K) & \max(0, Sd - K) \end{bmatrix} \begin{bmatrix} y_u \\ y_d \end{bmatrix} = \begin{bmatrix} S \\ B \\ O \end{bmatrix}$$

$$\begin{bmatrix} y_u \\ y_d \end{bmatrix} \geq 0$$

Black Scholes Formula

The first two equations can be solved for y_u and y_d:

$$\begin{bmatrix} y_u \\ y_d \end{bmatrix} = \begin{bmatrix} Su & Sd \\ Br & Br \end{bmatrix}^{-1} \begin{bmatrix} S \\ B \end{bmatrix} = \frac{1}{r(u-d)} \begin{bmatrix} r-d \\ u-r \end{bmatrix}$$

Then the last equation can be solved for O:

$$O = y_u \max(0, Su - K) + y_d \max(0, Sd - K)$$

This option pricing formula is the discrete analogue of the famous **Black-Scholes** formula.

Note:

- The nonnegativity requirement on y forces us to assume that $d < r < u$.

Probabilities

Suppose that one believes the up-scenario will happen with probability α and the down-scenario will happen with probability $\beta = 1 - \alpha$.

Then, one possible formula for the option price would be the expected present value of the option:

$$\mathbf{E}\frac{1}{r}\bar{S} = \alpha\frac{1}{r}\max(0, Su - K) + \beta\frac{1}{r}\max(0, Sd - K).$$

Here, $1/r$ is the discount factor.

Note that the Black-Scholes formula is of the same form but with specific formulas for α and β:

$$\begin{aligned}\alpha &= \frac{r - d}{u - d}\\ \beta &= \frac{u - r}{u - d}\end{aligned}$$

Many consider it a **feature** that the Black-Scholes formula does not depend on prespecified probabilities. In my opinion it is proof of a **bug** in the model. Which formula do you believe?