Linear Programming: Chapter 6
Matrix Notation

Robert J. Vanderbei

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Operations Research and Financial Engineering
Princeton University
Princeton, NJ 08544
http://www.princeton.edu/∼rvdb

An Example

Consider

$$\begin{array}{lrcrcrcl} \text{maximize} & 3x_1 & + & 4x_2 & - & 2x_3 & & \\ \text{subject to} & x_1 & + & 0.5x_2 & - & 5x_3 & \leq & 2 \\ & 2x_1 & - & x_2 & + & 3x_3 & \leq & 3 \\ & & & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

Add slacks (using x's for slack variables):

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 0.5x_2 & - & 5x_3 & + & x_4 & & & = & 2 \\ 2x_1 & - & x_2 & + & 3x_3 & & & + & x_5 & = & 3. \end{array}$$

Cast constraints into matrix notation:

$$\begin{bmatrix} 1 & 0.5 & -5 & 1 & 0 \\ 2 & -1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Similarly cast objective function:

$$\begin{bmatrix} 3 \\ 4 \\ -2 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

In general, we have:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Down the Road

Basic Variables: x_2, x_5.
Nonbasic Variables: x_1, x_3, x_4.

$$
\begin{aligned}
Ax &= \begin{bmatrix} x_1 + 0.5x_2 - 5x_3 + x_4 \\ 2x_1 - x_2 + 3x_3 + x_5 \end{bmatrix} \\
&= \begin{bmatrix} 0.5x_2 + x_1 - 5x_3 + x_4 \\ -x_2 + x_5 + 2x_1 + 3x_3 \end{bmatrix} \\
&= \begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 & -5 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} \\
&= Bx_{\mathcal{B}} + Nx_{\mathcal{N}}.
\end{aligned}
$$

General Matrix Notation

Up to a rearrangement of columns,

$$A \stackrel{\mathrm{R}}{=} \left[\begin{array}{cc} B & N \end{array}\right]$$

Similarly, rearrange rows of x and c:

$$x \stackrel{\mathrm{R}}{=} \left[\begin{array}{c} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{array}\right] \qquad c \stackrel{\mathrm{R}}{=} \left[\begin{array}{c} c_{\mathcal{B}} \\ c_{\mathcal{N}} \end{array}\right]$$

Constraints:

$$Ax = b \quad \Longleftrightarrow \quad Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b$$

Objective:

$$\zeta = c^T x \quad \Longleftrightarrow \quad c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}}$$

Matrix B is $m \times m$ and **invertible**! Why?

Express $x_{\mathcal{B}}$ and ζ in terms of $x_{\mathcal{N}}$:

$$\begin{aligned} x_{\mathcal{B}} &= B^{-1}b - B^{-1}Nx_{\mathcal{N}} \\ \zeta &= c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T B^{-1}b - \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right)^T x_{\mathcal{N}}. \end{aligned}$$

Dictionary in Matrix Notation

$$\begin{aligned} \zeta &= c_{\mathcal{B}}^T B^{-1}b \;-\; \left((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= \;\; B^{-1}b \;\;-\;\; B^{-1}Nx_{\mathcal{N}}. \end{aligned}$$

Example Revisited

$$B = \begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix} \quad \Longrightarrow \quad B^{-1} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$B^{-1}N = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & 1 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -10 & 2 \\ 4 & -7 & 2 \end{bmatrix}$$

$$(B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} = \begin{bmatrix} 2 & 4 \\ -10 & -7 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -38 \\ 8 \end{bmatrix}$$

$$c_{\mathcal{B}}^T B^{-1} b = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 16$$

Sanity Check

$$
\begin{array}{rcrcrcrcr}
\zeta & = & & & 3x_1 & + & 4x_2 & - & 2x_3 \\
\hline
x_4 & = & 2 & - & x_1 & - & 0.5x_2 & + & 5x_3 \\
x_5 & = & 3 & - & 2x_1 & + & x_2 & - & 3x_3.
\end{array}
$$

Let x_2 enter and x_4 leave.

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 16 & - & 5x_1 & - & 8x_4 & + & 38x_3 \\
\hline
x_2 & = & 4 & - & 2x_1 & - & 2x_4 & + & 10x_3 \\
x_5 & = & 7 & - & 4x_1 & - & 2x_4 & + & 7x_3.
\end{array}
$$

Dual Stuff

Associated Primal Solution:

$$\begin{aligned} x^*_{\mathcal{N}} &= 0 \\ x^*_{\mathcal{B}} &= B^{-1}b \end{aligned}$$

Dual Variables:

$$\begin{aligned} (x_1, \ldots, x_n, w_1, \ldots, w_m) &\longrightarrow (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \\ (z_1, \ldots, z_n, y_1, \ldots, y_m) &\longrightarrow (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) \end{aligned}$$

Associated Dual Solution:

$$\begin{aligned} z^*_{\mathcal{B}} &= 0 \\ z^*_{\mathcal{N}} &= (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} \end{aligned}$$

Associated Solution Value:

$$\zeta^* = c_{\mathcal{B}}^T B^{-1} b$$

Primal Dictionary:

$$\begin{aligned} \zeta &= \zeta^* - {z^*_{\mathcal{N}}}^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= x^*_{\mathcal{B}} - B^{-1} N x_{\mathcal{N}}. \end{aligned}$$

Dual Dictionary:

$$\begin{aligned} -\xi &= -\zeta^* - {x^*_{\mathcal{B}}}^T z_{\mathcal{B}} \\ z_{\mathcal{N}} &= z^*_{\mathcal{N}} + \left(B^{-1} N\right)^T z_{\mathcal{B}}. \end{aligned}$$

What have we gained?

1. A notation for doing proofs—no more proof by example.
2. Serious implementations of the simplex method avoid ever explicitly forming $B^{-1}N$. Reason:
   - The matrices B and N are sparse.
   - But B^{-1} is likely to be fully dense.
   - Even if B^{-1} is not dense, $B^{-1}N$ is going to be worse.
   - It's better simply to solve

     $$Bx_{\mathcal{B}} = b - Nx_{\mathcal{N}}$$

     efficiently.
   - This is subject of next chapter.
   - We'll skip it this year.

Primal Simplex

Suppose $x^*_{\mathcal{B}} \geq 0$
while ($z^*_{\mathcal{N}} \not\geq 0$) {

$$\text{pick } j \in \{j \in \mathcal{N} : z^*_j < 0\}$$

$$\Delta x_{\mathcal{B}} = B^{-1} N e_j$$

$$t = \left(\max_{i \in \mathcal{B}} \frac{\Delta x_i}{x^*_i} \right)^{-1}$$

$$\text{pick } i \in \operatorname{argmax}_{i \in \mathcal{B}} \frac{\Delta x_i}{x^*_i}$$

$$\Delta z_{\mathcal{N}} = -(B^{-1} N)^T e_i$$

$$s = \frac{z^*_j}{\Delta z_j}$$

$$x^*_j \leftarrow t, \qquad x^*_{\mathcal{B}} \leftarrow x^*_{\mathcal{B}} - t \Delta x_{\mathcal{B}}$$

$$z^*_i \leftarrow s, \qquad z^*_{\mathcal{N}} \leftarrow z^*_{\mathcal{N}} - s \Delta z_{\mathcal{N}}$$

$$\mathcal{B} \leftarrow \mathcal{B} \setminus \{i\} \cup \{j\}$$

}

Dual Simplex

Suppose $z^*_{\mathcal{N}} \geq 0$
while $(x^*_{\mathcal{B}} \not\geq 0)$ {

pick $i \in \{i \in \mathcal{B} : x^*_i < 0\}$

$\Delta z_{\mathcal{N}} = -(B^{-1}N)^T e_i$

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pick $j \in \operatorname{argmax}_{j \in \mathcal{N}} \dfrac{\Delta z_j}{z^*_j}$

$\Delta x_{\mathcal{B}} = B^{-1} N e_j$

$$t = \frac{x^*_i}{\Delta x_i}$$

$x^*_j \leftarrow t, \qquad x^*_{\mathcal{B}} \leftarrow x^*_{\mathcal{B}} - t\Delta x_{\mathcal{B}}$

$z^*_i \leftarrow s, \qquad z^*_{\mathcal{N}} \leftarrow z^*_{\mathcal{N}} - s\Delta z_{\mathcal{N}}$

$\mathcal{B} \leftarrow \mathcal{B} \setminus \{i\} \cup \{j\}$

}

Symmetry Lost

B is $m \times m$. Why not $n \times n$? What's go'in on?

A Problem and Its Dual

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \le b \\ & x \ge 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \ge c \\ & y \ge 0 \end{array}$$

Add Slacks

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax + w = b \\ & x, w \ge 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y - z = c \\ & y, z \ge 0 \end{array}$$

New Notations for Primal

$$\bar{A} = \begin{bmatrix} A & I \end{bmatrix}, \qquad \bar{c} = \begin{bmatrix} c \\ 0 \end{bmatrix}, \qquad \bar{x} = \begin{bmatrix} x \\ w \end{bmatrix}$$

New Notations for Dual

$$\hat{A} = \begin{bmatrix} -I & A^T \end{bmatrix}, \qquad \hat{b} = \begin{bmatrix} 0 \\ b \end{bmatrix}, \qquad \hat{z} = \begin{bmatrix} z \\ y \end{bmatrix}$$

Primal and Dual

$$\begin{array}{ll} \text{maximize} & \bar{c}^T \bar{x} \\ \text{subject to} & \bar{A}\bar{x} = b \\ & \bar{x} \geq 0 \end{array} \qquad\qquad \begin{array}{ll} \text{minimize} & \hat{b}^T \hat{z} \\ \text{subject to} & \hat{A}\hat{z} = c \\ & \hat{z} \geq 0 \end{array}$$

Symmetry Regained...

On the Primal Side:

$$\begin{bmatrix} A & I \end{bmatrix} \stackrel{\mathrm{R}}{=} \begin{bmatrix} \bar{N} & \bar{B} \end{bmatrix}$$

On the Dual Side:

$$\begin{bmatrix} -I & A^T \end{bmatrix} \stackrel{\mathrm{R}}{=} \begin{bmatrix} \hat{B} & \hat{N} \end{bmatrix}$$

Now Multiply:

$$\begin{aligned} \bar{A}\hat{A}^T &= \begin{bmatrix} \bar{N} & \bar{B} \end{bmatrix} \begin{bmatrix} \hat{B}^T \\ \hat{N}^T \end{bmatrix} \\ &= \bar{N}\hat{B}^T + \bar{B}\hat{N}^T \end{aligned}$$

And Again:

$$\begin{aligned} \bar{A}\hat{A}^T &= \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} -I \\ A \end{bmatrix} \\ &= -A + A = 0 \end{aligned}$$

The Two Expressions Must Be Equal:

$$\bar{N}\hat{B}^T + \bar{B}\hat{N}^T = 0$$

But That's the Negative Transpose Property:

$$\bar{B}^{-1}\bar{N} = -\left(\hat{B}^{-1}\hat{N}\right)^T$$