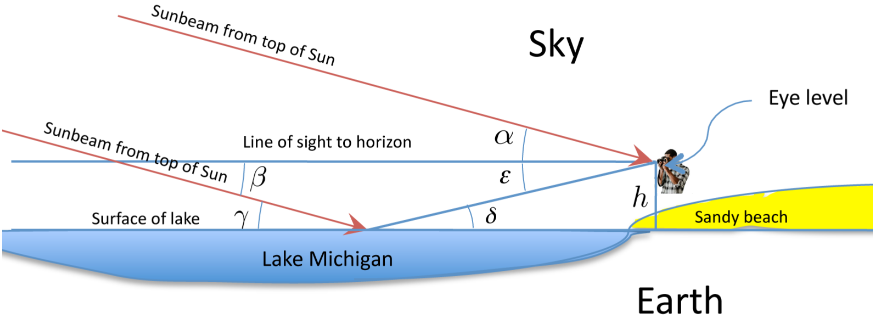
Geometry — If the Earth Were Flat!



$$
\begin{aligned}
\alpha &= \beta && \text{alternate interior angles are equal} \\
\beta &= \gamma && \text{alternate interior angles are equal} \\
\gamma &= \delta && \text{angle of incidence equals angle of reflection (from Physics!)} \\
\delta &= \epsilon && \text{alternate interior angles are equal}
\end{aligned}
$$

Therefore,

$$\alpha = \epsilon.$$

The reflection dips just as far below the horizon as the Sun stands above the horizon.

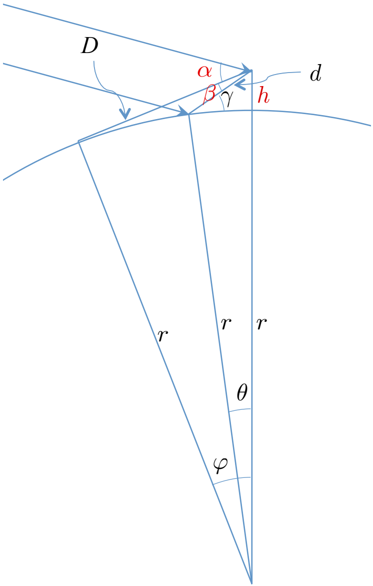
Geometry — The Earth Is Not Flat

Draw a picture.

Label everything of possible relevance.

Identify what we know:

- α Angle between horizon and top of Sun (measured from photo).
- β Angle between horizon and “top” of Sun in reflection (measured).
- h Height of “eye-level” above “water-level”.



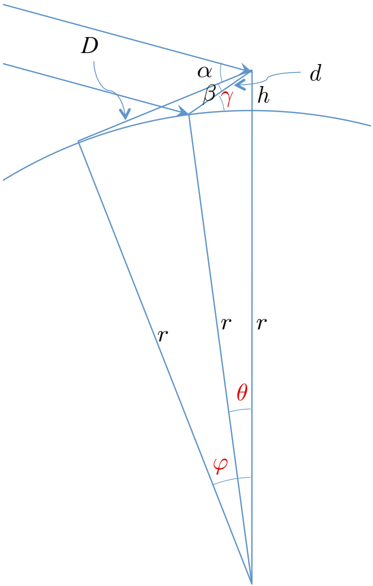
Geometry — The Earth Is Not Flat

Everything else is the stuff we need to figure out:

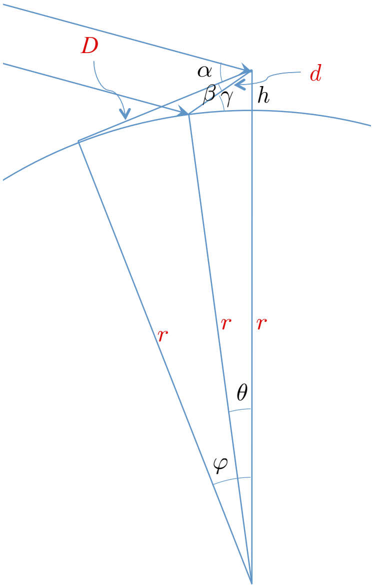
Three Angles:

- γ Angle of reflection off water.
- θ Angle between observer (me) and point of reflection.
- φ Angle between observer (me) and point of horizon.

Plus...



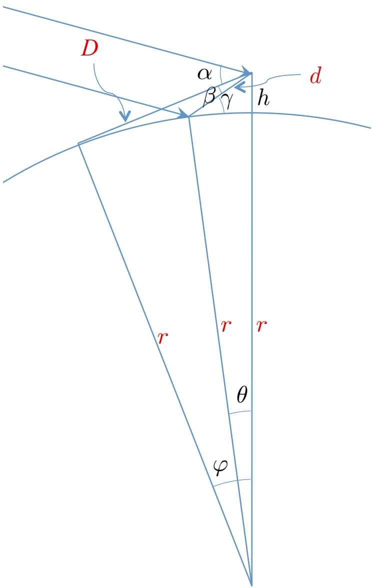
Geometry — The Earth Is Not Flat



Three Distances (lengths):

- *d* Distance to point of reflection.
- *D* Distance to horizon.
- *r* Radius of Earth.

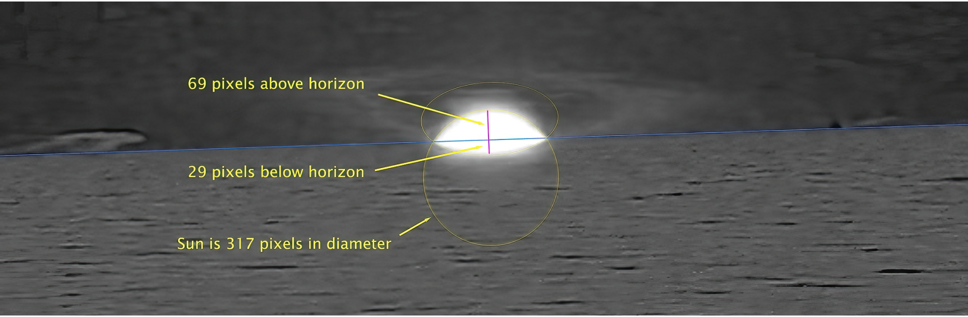
Geometry — The Earth Is Not Flat



Three Distances (lengths):

- *d* Distance to point of reflection.
- *D* Distance to horizon.
- *r* Radius of Earth. ⇐ This one is key

What We Know (Measure!)



The Sun is $1/2^\circ$ in diameter. Therefore, 1° equals $2 \times 317 = 634$ pixels. And so,

$$\alpha = 69 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.1088 \text{ degrees}$$

and

$$\beta = 29 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.0457 \text{ degrees.}$$

And, we assume that eye level is

$$h = 7 \text{ feet}$$

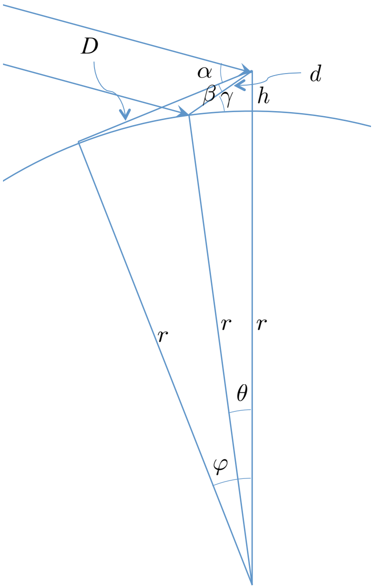
What We Need To Figure Out:

Angles:

- γ Angle of reflection off water.
- θ Angle between observer (me) and point of reflection.
- φ Angle between observer (me) and point of horizon.

Distances (lengths):

- d Distance to point of reflection.
- D Distance to horizon.
- r Radius of Earth.



What We Need To Figure Out:

Angles:

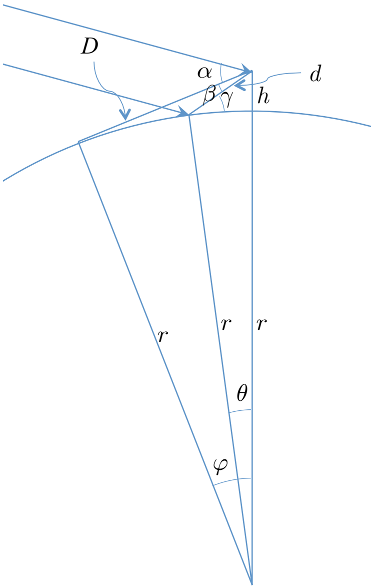
- γ Angle of reflection off water.
- θ Angle between observer (me) and point of reflection.
- φ Angle between observer (me) and point of horizon.

Distances (lengths):

- d Distance to point of reflection.
- D Distance to horizon.
- r Radius of Earth. ⇐ This one is key

That's SIX UNKNOWNS.

We need SIX EQUATIONS.



Equation 1:

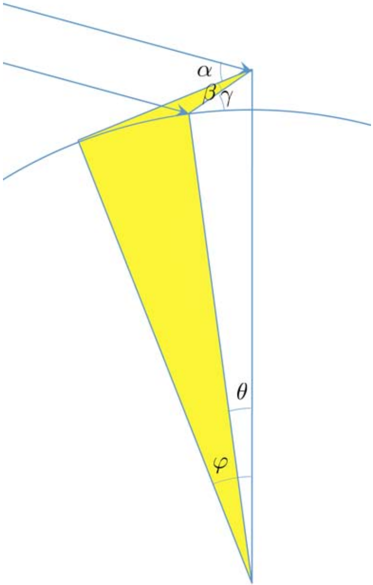
The sum of the angles around a quadrilateral is 360°.

Hence,

$$(\varphi - \theta) + 90 + \beta + (270 - \gamma) = 360.$$

Simplifying, we get

$$\varphi + \beta = \theta + \gamma.$$



Equation 2:

Given two parallel lines cut by a transversal, the consecutive interior angles are supplementary—they add up to 180°.
Hence,

$$\alpha + \beta + \sigma = 180.$$

Also, because angle of incidence equals angle of reflection, we see that

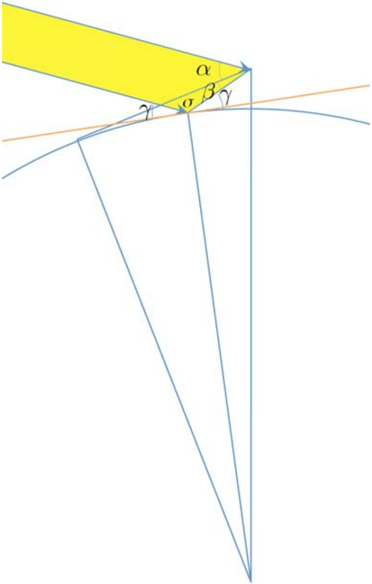
$$\gamma + \sigma + \gamma = 180.$$

Combining these two equations, we get

$$\alpha + \beta = 180 - \sigma = 2\gamma.$$

So, our second equation is

$$\alpha + \beta = 2\gamma.$$



Equation 3:

The distance from the center of the Earth to eye level is

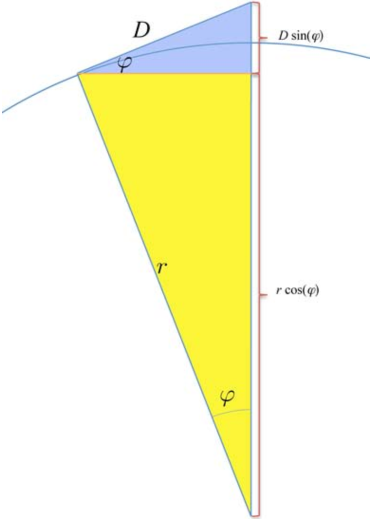
$$r + h.$$

But, it is also

$$D\sin(\varphi) + r\cos(\varphi).$$

Hence,

$$D\sin(\varphi) + r\cos(\varphi) = r + h.$$



Equation 4:

The "horizontal" distance from the point of the horizon on the water to the vertical line from the center of the Earth to eye level can be computed two ways:

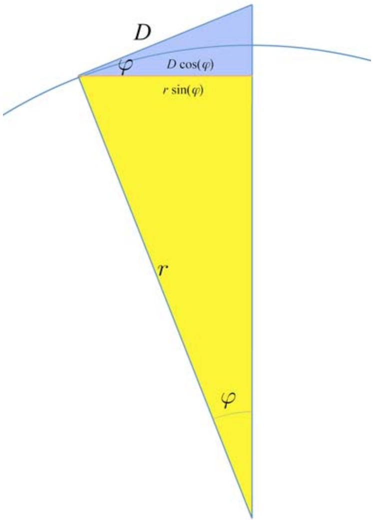
$$D \cos(\varphi)$$

and

$$r \sin(\varphi).$$

Hence,

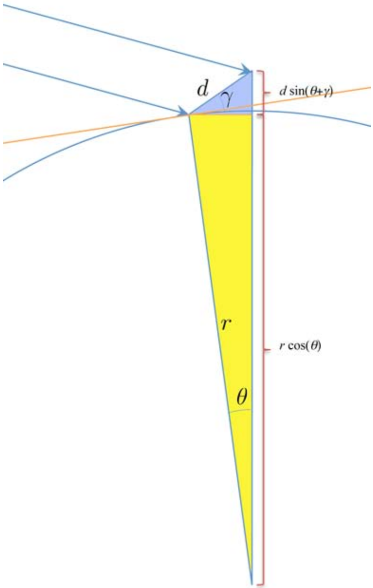
$$D \cos(\varphi) = r \sin(\varphi).$$



Equation 5:

Equation 5 is analogous to Equation 3, using the "point of reflection" in place of the "horizon".

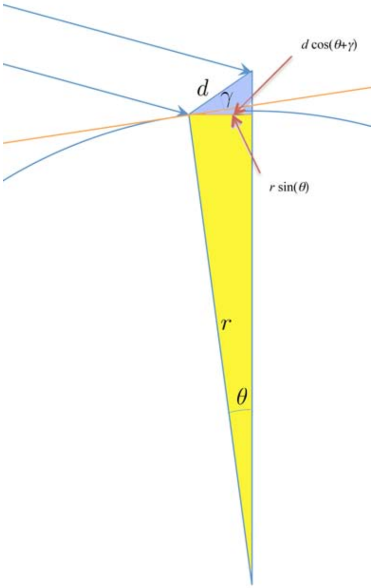
$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h.$$



Equation 6:

Equation 6 is analogous to Equation 4 in the same way.

$$d\cos(\theta+\gamma) = r\sin(\theta).$$



Six Equations in Six Unknowns:

$$\varphi + \beta = \theta + \gamma \tag{1}$$

$$\alpha + \beta = 2\gamma \tag{2}$$

$$D \sin(\varphi) + r \cos(\varphi) = r + h \tag{3}$$

$$D \cos(\varphi) = r \sin(\varphi) \tag{4}$$

$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h \tag{5}$$

$$d \cos(\theta + \gamma) = r \sin(\theta) \tag{6}$$

Six Equations in Six Unknowns:

$$\varphi + \beta = \theta + \gamma \tag{1}$$

$$\alpha + \beta = 2\gamma \tag{2}$$

$$D\sin(\varphi) + r\cos(\varphi) = r + h \tag{3}$$

$$D\cos(\varphi) = r\sin(\varphi) \tag{4}$$

$$d\sin(\theta + \gamma) + r\cos(\theta) = r + h \tag{5}$$

$$d\cos(\theta + \gamma) = r\sin(\theta) \tag{6}$$

Not hard to solve...

Six Equations in Six Unknowns:

$$\varphi + \beta = \theta + \gamma \tag{1}$$

$$\alpha + \beta = 2\gamma \tag{2}$$

$$D \sin(\varphi) + r \cos(\varphi) = r + h \tag{3}$$

$$D \cos(\varphi) = r \sin(\varphi) \tag{4}$$

$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h \tag{5}$$

$$d \cos(\theta + \gamma) = r \sin(\theta) \tag{6}$$

Not hard to solve...

Use (2) to solve for γ.

Six Equations in Six Unknowns:

$$\varphi + \beta = \theta + \gamma \tag{1}$$

$$\alpha + \beta = 2\gamma \tag{2}$$

$$D \sin(\varphi) + r \cos(\varphi) = r + h \tag{3}$$

$$D \cos(\varphi) = r \sin(\varphi) \tag{4}$$

$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h \tag{5}$$

$$d \cos(\theta + \gamma) = r \sin(\theta) \tag{6}$$

Not hard to solve...

Use (2) to solve for γ.
Solve (4) for D and then substitute in for D in (3).

Six Equations in Six Unknowns:

$$\varphi + \beta = \theta + \gamma \tag{1}$$

$$\alpha + \beta = 2\gamma \tag{2}$$

$$D \sin(\varphi) + r \cos(\varphi) = r + h \tag{3}$$

$$D \cos(\varphi) = r \sin(\varphi) \tag{4}$$

$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h \tag{5}$$

$$d \cos(\theta + \gamma) = r \sin(\theta) \tag{6}$$

Not hard to solve...

Use (2) to solve for γ.
Solve (4) for D and then substitute in for D in (3).
Solve (6) for d and then substitute in for d in (5).

Six Equations in Six Unknowns:

$$\varphi + \beta = \theta + \gamma \tag{1}$$

$$\alpha + \beta = 2\gamma \tag{2}$$

$$D \sin(\varphi) + r \cos(\varphi) = r + h \tag{3}$$

$$D \cos(\varphi) = r \sin(\varphi) \tag{4}$$

$$d \sin(\theta + \gamma) + r \cos(\theta) = r + h \tag{5}$$

$$d \cos(\theta + \gamma) = r \sin(\theta) \tag{6}$$

Not hard to solve...

Use (2) to solve for γ.
Solve (4) for D and then substitute in for D in (3).
Solve (6) for d and then substitute in for d in (5).
Use (1) to change $\theta + \gamma$ to $\varphi + \beta$ in (5) and (6).

Six Equations in Six Unknowns:

$$\begin{aligned}
\varphi + \beta &= \theta + \gamma && (1) \\
\alpha + \beta &= 2\gamma && (2) \\
D \sin(\varphi) + r \cos(\varphi) &= r + h && (3) \\
D \cos(\varphi) &= r \sin(\varphi) && (4) \\
d \sin(\theta + \gamma) + r \cos(\theta) &= r + h && (5) \\
d \cos(\theta + \gamma) &= r \sin(\theta) && (6)
\end{aligned}$$

Not hard to solve...

Use (2) to solve for γ.
Solve (4) for D and then substitute in for D in (3).
Solve (6) for d and then substitute in for d in (5).
Use (1) to change $\theta + \gamma$ to $\varphi + \beta$ in (5) and (6).
And so on...