Geometry — If the Earth Were Flat!

\[ \alpha = \beta \] alternate interior angles are equal
\[ \beta = \gamma \] alternate interior angles are equal
\[ \gamma = \delta \] angle of incidence equals angle of reflection (from Physics!)
\[ \delta = \varepsilon \] alternate interior angles are equal

Therefore,

\[ \alpha = \varepsilon. \]

The reflection dips just as far below the horizon as the Sun stands above the horizon.
Geometry — The Earth Is Not Flat

Draw a picture.

Label everything of possible relevance.

Identify what we know:

- $\alpha$ Angle between horizon and top of Sun (measured from photo).
- $\beta$ Angle between horizon and “top” of Sun in reflection (measured).
- $h$ Height of “eye-level” above “water-level”.

\[ D \]

\[ \alpha \]

\[ \beta \]

\[ h \]
Geometry — The Earth Is Not Flat

Everything else is the stuff we need to figure out:

Three Angles:

\( \gamma \) Angle of reflection off water.
\( \theta \) Angle between observer (me) and point of reflection.
\( \varphi \) Angle between observer (me) and point of horizon.

Plus...
Geometry — The Earth Is Not Flat

Three Distances (lengths):

- $d$ Distance to point of reflection.
- $D$ Distance to horizon.
- $r$ Radius of Earth.
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- $d$: Distance to point of reflection.
- $D$: Distance to horizon.
- $r$: Radius of Earth. $\Leftarrow$ This one is key
The Sun is 1/2° in diameter. Therefore, 1° equals 2 × 317 = 634 pixels. And so,

\[ \alpha = 69 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.1088 \text{ degrees} \]

and

\[ \beta = 29 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.0457 \text{ degrees} \]

And, we assume that eye level is

\[ h = 7 \text{ feet} \]
What We Need To Figure Out:

Angles:
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- $\theta$ Angle between observer (me) and point of reflection.
- $\varphi$ Angle between observer (me) and point of horizon.

Distances (lengths):
- $d$ Distance to point of reflection.
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That's SIX UNKNOWNS.
We need SIX EQUATIONS.
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That’s SIX UNKNOWNS.

We need SIX EQUATIONS.
Equation 1:

The sum of the angles around a quadrilateral is 360°.

Hence,

\[(\varphi - \theta) + 90 + \beta + (270 - \gamma) = 360.\]

Simplifying, we get

\[\varphi + \beta = \theta + \gamma.\]
Equation 2:

Given two parallel lines cut by a transversal, the consecutive interior angles are supplementary—they add up to $180^\circ$. Hence,

$$\alpha + \beta + \sigma = 180.$$ 

Also, because angle of incidence equals angle of reflection, we see that

$$\gamma + \sigma + \gamma = 180.$$ 

Combining these two equations, we get

$$\alpha + \beta = 180 - \sigma = 2\gamma.$$ 

So, our second equation is

$$\alpha + \beta = 2\gamma.$$
Equation 3:
The distance from the center of the Earth to eye level is

\[ r + h. \]

But, it is also

\[ D \sin(\varphi) + r \cos(\varphi). \]

Hence,

\[ D \sin(\varphi) + r \cos(\varphi) = r + h. \]
Equation 4:

The “horizontal” distance from the point of the horizon on the water to the vertical line from the center of the Earth to eye level can be computed two ways:

\[ D \cos(\varphi) \]

and

\[ r \sin(\varphi). \]

Hence,

\[ D \cos(\varphi) = r \sin(\varphi). \]
Equation 5:

Equation 5 is analogous to Equation 3, using the “point of reflection” in place of the “horizon”.

\[ d \sin(\theta + \gamma) + r \cos(\theta) = r + h. \]
Equation 6 is analogous to Equation 4 in the same way.

\[ d \cos(\theta + \gamma) = r \sin(\theta). \]
Six Equations in Six Unknowns:

\[
\varphi + \beta = \theta + \gamma \quad (1)
\]

\[
\alpha + \beta = 2\gamma \quad (2)
\]

\[
D \sin(\varphi) + r \cos(\varphi) = r + h \quad (3)
\]

\[
D \cos(\varphi) = r \sin(\varphi) \quad (4)
\]

\[
d \sin(\theta + \gamma) + r \cos(\theta) = r + h \quad (5)
\]

\[
d \cos(\theta + \gamma) = r \sin(\theta) \quad (6)
\]

Not hard to solve...

Use (2) to solve for \( \gamma \).

Solve (4) for \( D \) and then substitute in for \( D \) in (3).

Solve (6) for \( d \) and then substitute in for \( d \) in (5).

Use (1) to change \( \theta + \gamma \) to \( \varphi + \beta \) in (5) and (6).

And so on...
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