Olbers' Paradox

Freshman Seminar 131

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Olbers pointed out that...

If the stars that we see in the night sky are all similar to our own Sun

and if they are randomly distributed in space

and extend without limit to infinity,

then the night sky would be blazingly bright (brighter than our current daytime sky)

because, no matter what direction we look, we will be looking directly at a star.

And, distance doesn't make a star's "surface brightness" any fainter.

Double the distance, and the total amount of light goes down by a factor of 4.

But, the size goes down by a factor of 4 too.

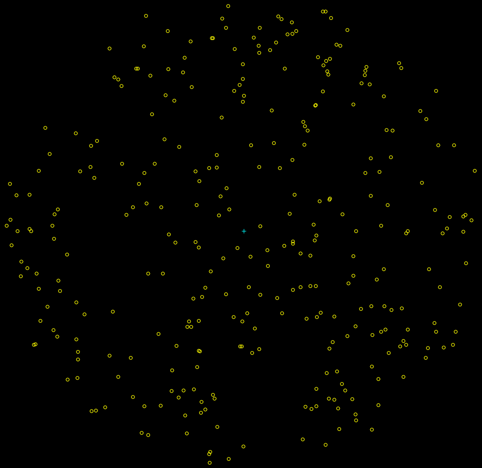
The surface brightness, i.e., amount of light per unit area, remains unchanged.

So, can we estimate how big the universe is by comparing the “brightness” of the night sky to the surface brightness of the Sun?

The answer is: YES.

But, we’ll need to derive some formulas...

Let's first work out the 2D case.
Suppose for simplicity that our 2D universe has radius R.
Suppose we are at the center of this universe.
Suppose that stars, of radius ρ (we assume ρ is much much smaller than R), are randomly distributed throughout the universe...



Let's do a little bit of calculus (or just jump to the next slide)...

Let's imagine we are looking from the center of the universe (the blue '+') along the positive x-axis.

Let's start by considering just a single star of radius ρ.

Let (x, y) denote the coordinates of this star.

Let (r, θ) denote it's polar coordinates ($r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$).

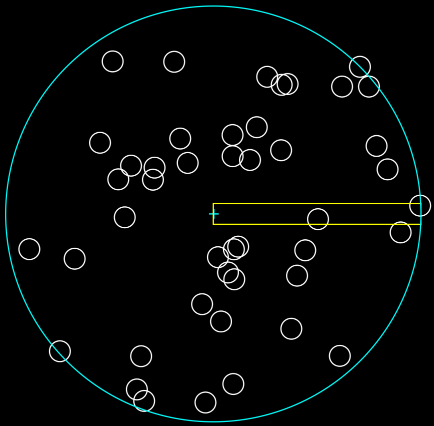
This star will block our x-axis view if and only if

$$|\theta| \leq \frac{\rho}{r}$$

Averaging over all possible locations for this star, we can compute the probability that this single star will block our view:

$$\begin{aligned} \text{Prob(star blocks view)} &= \int_0^R \int_{|\theta| \leq \frac{\rho}{r}} \frac{1}{\pi R^2} r d\theta dr \\ &= \frac{2\rho}{\pi R} \end{aligned}$$

Here's a geometric derivation.
A star will block our view along the positive x-axis if and only if the center of the star lies in this blue rectangle:



The area of the rectangle is $2\rho R$. The area inside the circle is πR^2.
The ratio of these areas, $2\rho/\pi R$, is the probability that a single star will block our view.

Now, let's do the same analysis but in 3D.

The rectangle becomes a cylinder whose base has radius ρ and whose "height" is R.

Hence, the blocking volume is $\pi\rho^2 R$.

And, the volume of the sphere is $\frac{4}{3}\pi R^3$.

The probability that a single star blocks our view is just the ratio:

$$\text{Prob(Star blocks view)} = \frac{3\rho^2}{4R^2}$$

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The probability that a single star blocks our view is just the ratio:

$$\text{Prob(Star blocks view)} = \frac{3\rho^2}{4R^2}$$

Sanity Check: a probability is a number between 0 and 1. It is *unitless*. That is, it is not a length or a mass or a time. These things have units. Let's check the formula. If ρ is measured in meters, then the numerator has units of meters-squared. If R in the denominator is also measured in meters, then the ratio is meters-squared over meters-squared. In other words... unitless.

Now, let's consider lots of stars. Let's say there are n stars.

It's tricky to compute the probability that *at least one of the stars blocks our view*.

It's much easier to compute the probability that *none of the stars block our view*.

Assuming that the stars locations are *independently* determined, then the probability that none of the stars block our view is just the n-th power of the probability that any one of the stars does not block our view:

$$\text{Prob}(\text{None of the } n \text{ stars block our view}) = \left(1 - \frac{3\rho^2}{4R^2}\right)^n$$

And so the probability of at least one star blocking our view is

$$\text{Prob}(\text{At least one of the } n \text{ stars blocks our view}) = 1 - \left(1 - \frac{3\rho^2}{4R^2}\right)^n$$

Using the binomial expansion (aka binomial theorem),

$$(a+b)^n = a^n + na^{n-1}b + \begin{pmatrix} n \\ 2 \end{pmatrix} a^{n-2}b^2 + \cdots + \begin{pmatrix} n \\ j \end{pmatrix} a^{n-j}b^j + \cdots + b^n$$

together with our assumption that ρ is *much smaller* than R, we get this simple approximation:

$$\begin{aligned} \text{Prob(At least one blocking star)} &= 1 - \left(1 - \frac{3\rho^2}{4R^2}\right)^n \\ &= 1 - 1 + n\frac{3\rho^2}{4R^2} - \begin{pmatrix} n \\ 2 \end{pmatrix} \left(\frac{3\rho^2}{4R^2}\right)^2 + \cdots \\ &\approx n\frac{3\rho^2}{4R^2} \end{aligned}$$

Using the binomial expansion (aka binomial theorem),

$$(a+b)^n = a^n + na^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{j} a^{n-j}b^j + \cdots + b^n$$

together with our assumption that ρ is *much smaller* than R, we get this simple approximation:

$$\begin{aligned}
\text{Prob(At least one blocking star)} &= 1 - \left(1 - \frac{3\rho^2}{4R^2}\right)^n \\
&= 1 - 1 + n\frac{3\rho^2}{4R^2} - \binom{n}{2}\left(\frac{3\rho^2}{4R^2}\right)^2 + \cdots \\
&\approx n\frac{3\rho^2}{4R^2}
\end{aligned}$$

Note: This formula is just n times the one-star formula. In other words, the probability seems additive. This would be exactly correct if the stars never overlap. Overlapping is possible, but it is highly highly highly unlikely and so this approximation is a good one.

Let p denote the probability of at least one star in our line of sight. From the previous slide, we have

$$p = n\frac{3\rho^2}{4R^2}$$

We can measure ρ (it's the radius of a typical star, such as our own Sun).

But, n and R are related to each other. We need to quantify this relation.

Let d denote the average distances between "nearest neighbor" pairs of stars. We know that $d \approx 4$ light years.

It's tricky to compute roughly how many stars, n, there are in a sphere of radius R. Let's first consider a 3D lattice and assume that there is a star at each lattice point. If the lattice has width $2R$, then the cubic lattice would contain $(2R/d)^3$ stars. But, our sphere is a subset of this cube and so we need to scale this down by the ratio of the volume of a sphere to that of a cube:

$$n = \left(\frac{2R}{d}\right)^3 \frac{\frac{4}{3}\pi R^3}{(2R)^3} = \frac{4\pi R^3}{3d^3}.$$

If we plug this formula for n into our formula for p, we get

$$p = \frac{4\pi R^3}{3d^3}\,\frac{3\rho^2}{4R^2} = \frac{\pi R\rho^2}{d^3}.$$

We're almost done.
Recall the formula from the previous slide:

$$p = \frac{\pi R \rho^2}{d^3}.$$

We know d and ρ and we can measure p. So, we can use this formula to compute R:

$$R = \frac{pd^3}{\pi \rho^2}.$$

It's convenient to rewrite this formula as follows:

$$R = p\,d\left(\frac{d^2}{\pi \rho^2}\right).$$

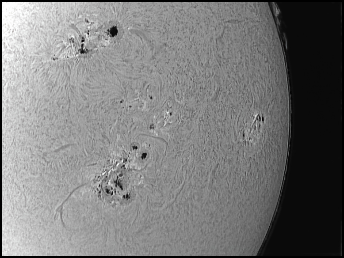
We (will eventually) know that $d \approx 253000\mathrm{AU} \approx 1.23$ parsecs ≈ 4 ly and $\rho = 0.00465$ AU and so we get

$$\begin{aligned} R &= p\,d\left(9.42 \times 10^{14}\right) \\ &= p \times \left(3.77 \times 10^{15}\right) \text{ light years .} \end{aligned}$$

The constant is *huge*. But, p is going to be small. Let's go out and measure it!

Questar telescope with 3x focal reducer.
MX-916 camera with L, R, G, B filters.
Average pixel brightness value: 2
Exposure time: 6 min
Time to fullness:

$$6 \text{ min} \times (255/2) \times 3^2 = 6886 \text{ min} = 413100 \text{ secs}$$



Questar telescope
MX-916 camera with Coronado ASP60 Hα filter.
Filter bandpass is 0.07nm.
Visible spectrum spans about 350nm.
Ratio is: 0.07nm/350nm = 0.0002
And "Moon" filter. Updated ratio: 0.00002
Exposure time: 0.3 seconds.
Without filter, exposure time would be:

$$0.3 \text{ secs} \times 0.00002 = 0.000006 \text{ secs}$$

$$p = 0.000006/413100 = 1.4 \times 10^{-11}$$

$$R = p \cdot (3.77 \times 10^{15}) = (1.4 \times 10^{-11}) \cdot (3.77 \times 10^{15}) = 51800 \text{ ly}$$

10″ Ritchey-Chretien and Trius Camera

Location	**Date**	**Exp. (sec)**	**Photons/pix**	**Photons/pix/sec**	p	R **(ly)**
Blue Sky Baader-Hα	21-06-18	600	2842	4.73		
Blue Sky Luminance	21-06-18	0.001	23700	2.37×10^7		
Sun Baader-Hα	21-06-18	0.03	42500	1.42×10^6		
Sun Luminance				7.11×10^{12}		
M38	20-03-15	20	16.6	0.83	1.2×10^{-13}	439
M44	21-03-20	20	356	17.80	2.6×10^{-12}	9639
M52	20-12-11	10	226	22.60	3.1×10^{-12}	11777
M67	20-04-18	40	306	7.65	1.1×10^{-12}	4068
NGC 1342	20-11-27	20	163	8.15	1.2×10^{-12}	4335
NGC 7062	21-09-19	10	360	36.00	5.1×10^{-12}	19272
Vanderbei Star	18-01-09	14	14.2	1.01	1.4×10^{-13}	535