Olbers’ Paradox

Freshman Seminar 131

Robert J. Vanderbei

2020
Olbers pointed out that...

If the stars that we see in the night sky are all similar to our own Sun and if they are randomly distributed in space and extend without limit to infinity, then the night sky would be blazingly bright (brighter than our current daytime sky) because, no matter what direction we look, we will be looking directly at a star.

And, distance doesn’t make a star’s “surface brightness” any fainter. Double the distance, and the total amount of light goes down by a factor of 4. But, the size goes down by a factor of 4 too. The surface brightness, i.e., amount of light per unit area, remains unchanged.
So, can we estimate how big the universe is by comparing the “brightness” of the night sky to the surface brightness of the Sun?

The answer is: YES.

But, we’ll need to derive some formulas...
Let’s first work out the 2D case.

Suppose for simplicity that our 2D universe has radius $R$.

Suppose we are at the center of this universe.

Suppose that stars, of radius $\rho$ (we assume $\rho$ is much much smaller than $R$), are randomly distributed throughout the universe...
Let’s do a little bit of calculus (or just jump to the next slide)...

Let’s imagine we are looking from the center of the universe (the red ‘+’) along the positive $x$-axis.

Let’s start by considering just a single star of radius $\rho$.

Let $(x, y)$ denote the coordinates of this star.

Let $(r, \theta)$ denote it’s polar coordinates ($r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$).

This star will block our $x$-axis view if and only if

$$|\theta| \leq \frac{\rho}{r}$$

Averaging over all possible locations for this star, we can compute the probability that this single star will block our view:

$$\text{Prob}(\text{star blocks view}) = \int_0^R \int_{|\theta| \leq \frac{\rho}{r}} \frac{1}{\pi R^2} r \,d\theta \,dr$$

$$= \frac{2\rho}{\pi R}$$
Here’s a geometric derivation.
A star will block our view along the positive $x$-axis if and only if the center of the star lies in this blue rectangle:

The area of the rectangle is $2\rho R$. The area inside the circle is $\pi R^2$. The ratio of these areas, $2\rho / \pi R$, is the probability that a single star will block our view.
Discussion Question:

The calculus derivation was *exact*, whereas the geometry derivation was approximate (the right-hand side of the rectangle is a little curved).

But, they gave exactly the same answer.

WHY?
Discussion Question:

The calculus derivation was *exact*, whereas the geometry derivation was approximate (the right-hand side of the rectangle is a little curved).

But, they gave exactly the same answer.

***Why?***

*Hint:* The calculus derivation is not exact. Think about it.
Now, let’s do the same analysis but in 3D.

The rectangle becomes a cylinder whose base has radius $\rho$ and whose “height” is $R$.

Hence, the blocking volume is $\pi \rho^2 R$.

And, the volume of the sphere is $\frac{4}{3} \pi R^3$.

The probability that a single star blocks our view is just the ratio:

$$\text{Prob(Star blocks view)} = \frac{3 \rho^2}{4 R^2}$$

_Sanity Check:_ a probability is a number between 0 and 1. It is _unitless_. That is, it is not a length or a mass or a time. These things have units. Let’s check the formula. If $\rho$ is measured in meters, then the numerator has units of meters-squared. If $R$ in the denominator is also measured in meters, then the ratio is meters-squared over meters-squared. In other words... unitless.
Now, let’s consider lots of stars. Let’s say there are \( n \) stars.

It’s tricky to compute the probability that \textit{at least one of the stars blocks our view}.

It’s much easier to compute the probability that \textit{none of the stars block our view}.

Assuming that the stars locations are \textit{independently} determined, then the probability that none of the stars block our view is just the \( n \)-th power of the probability that any one of the stars does not block our view:

\[
\text{Prob}(\text{None of the } n \text{ stars block our view}) = \left(1 - \frac{3\rho^2}{4R^2}\right)^n
\]

And so the probability of at least one star blocking our view is

\[
\text{Prob}(\text{At least one of the } n \text{ stars blocks our view}) = 1 - \left(1 - \frac{3\rho^2}{4R^2}\right)^n
\]
Using the binomial expansion,

\[(a + b)^n = a^n + n a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{j} a^{n-j} b^j + \cdots + b^n\]

together with our assumption that \(\rho\) is much smaller than \(R\), we get this simple approximation:

\[
\text{Prob(At least one blocking star)} = 1 - \left(1 - \frac{3\rho^2}{4R^2}\right)^n
\]

\[
= 1 - 1 + n \frac{3\rho^2}{4R^2} - \binom{n}{2} \left(\frac{3\rho^2}{4R^2}\right)^2 + \cdots
\]

\[
\approx n \frac{3\rho^2}{4R^2}
\]

**Note:** This formula is just \(n\) times the one-star formula. In other words, the probability seems additive. This would be exactly correct if the stars never overlap. Overlapping is possible, but it is highly highly highly unlikely and so this approximation is a good one.
Let \( p \) denote the probability of at least one star in our line of sight. From the previous slide, we have

\[
p = n \frac{3 \rho^2}{4R^2}
\]

We can measure \( \rho \) (it’s the radius of a typical star, such as our own Sun).

But, \( n \) and \( R \) are related to each other. We need to quantify this relation.

Let \( d \) denote the average distances between “nearest neighbor” pairs of stars. We know that \( d \approx 4 \) light years.

It’s tricky to compute roughly how many stars, \( n \), there are in a sphere of radius \( R \). Let’s first consider a 3D lattice and assume that there is a star at each lattice point. If the lattice has width \( 2R \), then the cubic lattice would contain \((2R/d)^3\) stars. But, our sphere is a subset of this cube and so we need to scale this down by the ratio of the volume of a sphere to that of a cube:

\[
n = \left( \frac{2R}{d} \right)^3 \frac{4\pi R^3}{3} = \frac{4\pi R^3}{3d}\frac{3}{d^3} = \frac{4\pi R^3}{3d^3}.
\]

If we plug this formula for \( n \) into our formula for \( p \), we get

\[
p = \frac{4\pi R^3}{3d^3} \frac{3 \rho^2}{4R^2} = \frac{\pi R \rho^2}{d^3}.
\]
We’re almost done. 
Recall the formula from the previous slide:

\[ p = \frac{\pi R \rho^2}{d^3}. \]

We know \( d \) and \( \rho \) and we can measure \( p \). So, we can use this formula to compute \( R \):

\[ R = \frac{pd^3}{\pi \rho^2}. \]

It’s convenient to rewrite this formula as follows:

\[ R = pd \left( \frac{d^2}{\pi \rho^2} \right). \]

We know that \( d \approx 4 \) light years and \( \rho = 432,000 \) miles \( \approx 7.33535 \times 10^{-8} \) light years and so we get

\[
R = pd \left(9.42 \times 10^{14}\right)
\]

\[ = pd \times \left(3.77 \times 10^{15}\right) \text{ light years}. \]

The constant is huge. But, \( p \) is going to be small. Let’s go out and measure it!
Questar telescope with 3x focal reducer.
Average pixel brightness value: 2
Exposure time: 6 min
Time to fullness:
\[ 6 \text{ min} \times (255/2) \times 3^2 = 6886 \text{ min} = 413100 \text{ secs} \]

Questar telescope
MX-916 camera with Coronado ASP60 Hα filter.
Filter bandpass is 0.7 Å.
Visible spectrum spans about 3500 Å.
Ratio is: \( 0.7\text{Å}/3500\text{Å} = 0.0002 \)
And “Moon” filter. Updated ratio: 0.00002
Exposure time: 0.3 seconds.
Without filter, exposure time would be:
\[ 0.3 \text{ secs} \times 0.00002 = 0.000006 \text{ secs} \]

\[ p = 0.000006/413100 = 1.4 \times 10^{-11} \]
\[ R = p \cdot (3.77 \times 10^{15}) = (1.4 \times 10^{-11}) \cdot (3.77 \times 10^{15}) = 51800 \text{ ly} \]
<table>
<thead>
<tr>
<th>Location</th>
<th>Date</th>
<th>Exp. (sec)</th>
<th>Photons/pix</th>
<th>Photons/pix/sec</th>
<th>$p$</th>
<th>$R$ (ly)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue Sky Baader-Hα</td>
<td>21-06-18</td>
<td>600</td>
<td>2000</td>
<td>3.33</td>
<td>2.37 × 10^7</td>
<td></td>
</tr>
<tr>
<td>Blue Sky Luminance</td>
<td>21-06-18</td>
<td>0.001</td>
<td>23700</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sun Baader-Hα</td>
<td>21-06-18</td>
<td>0.03</td>
<td>42500</td>
<td>1.42 × 10^6</td>
<td>1.01 × 10^13</td>
<td></td>
</tr>
<tr>
<td>Sun Luminance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M38</td>
<td>20-03-15</td>
<td>20</td>
<td>16.6</td>
<td>0.83</td>
<td>8.2 × 10^{-14}</td>
<td>309</td>
</tr>
<tr>
<td>M44</td>
<td>21-03-20</td>
<td>20</td>
<td>356</td>
<td>17.80</td>
<td>1.8 × 10^{-12}</td>
<td>6786</td>
</tr>
<tr>
<td>M52</td>
<td>20-12-11</td>
<td>10</td>
<td>226</td>
<td>22.60</td>
<td>2.2 × 10^{-12}</td>
<td>8294</td>
</tr>
<tr>
<td>M67</td>
<td>20-04-18</td>
<td>40</td>
<td>306</td>
<td>7.65</td>
<td>7.6 × 10^{-13}</td>
<td>2865</td>
</tr>
<tr>
<td>NGC 1342</td>
<td>20-11-27</td>
<td>20</td>
<td>163</td>
<td>8.15</td>
<td>8.1 × 10^{-13}</td>
<td>3053</td>
</tr>
<tr>
<td>NGC 7062</td>
<td>21-09-19</td>
<td>10</td>
<td>360</td>
<td>36.00</td>
<td>3.6 × 10^{-12}</td>
<td>13572</td>
</tr>
<tr>
<td>Vanderbei Star</td>
<td>18-01-09</td>
<td>14</td>
<td>14.2</td>
<td>1.01</td>
<td>1.0 × 10^{-13}</td>
<td>377</td>
</tr>
</tbody>
</table>