The Earth Is Not Flat
An Analysis of a Sunset Photo

Can a photo of the sunset over Lake Michigan reveal the shape of our planet?
I will show you how we can...

measure something \textit{BIG} (the size of the Earth)

by first measuring something \textit{small} (my height), and measuring an \textit{angle} (off from a photograph)

and then doing some \textit{geometry}.
The Earth is a big sphere. How do we know?

Several ways. One way is to look at a Lunar eclipse...

Photo taken March 3, 2007, at about 8pm.
From a lunar eclipse, we can determine that the Earth is about 3 or 4 times larger than the Moon. But, how big is the Earth?

Next total lunar eclipse visible from the “east coast” is on January 20/21, 2019.
How big is the Earth? How can we find out?

First Method: Look it up on Wikipedia.

You’ll get the right answer (radius = 3,960 miles), but no satisfaction.

Second Method: Air travel.

I’ve flown to Bangkok Thailand.  
It’s about a 17 hour flight.  
It’s about halfway around the Earth.  
Jets fly at about 600 mph.  
So, the distance I flew is \( \text{about} \)

\[ 17 \text{ hours} \times 600 \text{ miles/hour} = 10,200 \text{ miles} \]

The circumference is then about 20,000 miles and radius is therefore about \( 20,000/2\pi = 3,250 \text{ miles} \). This is just a rough estimate.
IS THERE AN EASIER WAY?
A picture I took of a sunset over Lake Michigan.
A close-up.

Using this picture, some geometry, and a little trigonometry, I was able to compute that the Earth’s radius is about 5000 miles.
A smooth lake is supposed to act like a mirror.
The Sun’s reflection should have looked something like this...
Or not!

What’s going on?
Lake Michigan is not a flat mirror.

Its surface is curved because the Earth is a sphere.

That’s why we can’t see the shore on the opposite side—it’s below the horizon!
Geometry — If the Earth Were Flat!

\[ \alpha = \beta \] alternate interior angles are equal
\[ \beta = \gamma \] alternate interior angles are equal
\[ \gamma = \delta \] angle of incidence equals angle of reflection (from Physics!)
\[ \delta = \epsilon \] alternate interior angles are equal

Therefore,
\[ \alpha = \epsilon. \]

The reflection dips just as far below the horizon as the Sun stands above the horizon.
Geometry — The Earth Is Not Flat

Draw a picture.

Label everything of possible relevance.

Identify what we know:

\[ \alpha \] Angle between horizon and top of Sun (measured from photo)

\[ \beta \] Angle between horizon and “top” of Sun in reflection (measured)

\[ h \] Height of “eye-level” above “water-level”.

Geometry — The Earth Is Not Flat

Everything else is the stuff we need to figure out:

- Three Angles:
  - $\gamma$: Angle of reflection off water.
  - $\theta$: Angle between observer (me) and point of reflection.
  - $\varphi$: Angle between observer (me) and point of horizon.

Plus...
Geometry — The Earth Is Not Flat

- Three Distances (lengths):
  - $d$: Distance to point of reflection.
  - $D$: Distance to horizon.
  - $r$: Radius of Earth. $\leftarrow$ This one is key!!!
The Sun is $1/2^\circ$ in diameter. Therefore, $1^\circ$ equals $2 \times 317 = 634$ pixels.

And so,

$$\alpha = 69 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.1088 \text{ degrees}$$

and

$$\beta = 29 \text{ pixels} \times \frac{1 \text{ degree}}{634 \text{ pixels}} = 0.0457 \text{ degrees}.$$ 

And, we assume that eye level is

$$h = 7 \text{ feet}$$
What We Need To Figure Out:

- **Angles:**
  - $\gamma$ Angle of reflection off water.
  - $\theta$ Angle between observer (me) and point of reflection.
  - $\varphi$ Angle between observer (me) and point of horizon.

- **Distances (lengths):**
  - $d$ Distance to point of reflection.
  - $D$ Distance to horizon.
  - $r$ Radius of Earth. \(\Leftarrow\) This one is key!!!

That’s SIX UNKNOWNS.

We need SIX (distinct!) EQUATIONS.
Equation 1:

The sum of the angles around a quadrilateral is 360°.

Hence,

\[(\varphi - \theta) + 90 + \beta + (270 - \gamma) = 360.\]

Simplifying, we get

\[\varphi + \beta = \theta + \gamma.\]
Equation 2:

Given two parallel lines cut by a transversal, the consecutive interior angles are supplementary—they add up to $180^\circ$.

Hence,

$$\alpha + \beta + \sigma = 180.$$ 

Also, because angle of incidence equals angle of reflection, we see that

$$\gamma + \sigma + \gamma = 180.$$ 

Combining these two equations, we get

$$\alpha + \beta = 180 - \sigma = 2\gamma.$$ 

So, our second equation is

$$\alpha + \beta = 2\gamma.$$
Equation 3:

The distance from the center of the Earth to eye level is

$$r + h.$$  

But, it is also

$$D \sin(\varphi) + r \cos(\varphi).$$

Hence,

$$D \sin(\varphi) + r \cos(\varphi) = r + h.$$
Equation 4:

The “horizontal” distance from the point of the horizon on the water to the vertical line from the center of the Earth to eye level can be computed two ways:

\[ D \cos(\varphi) \]

and

\[ r \sin(\varphi). \]

Hence,

\[ D \cos(\varphi) = r \sin(\varphi). \]
Equation 5:

Equation 5 is analogous to Equation 3, using the “point of reflection” in place of the “horizon”.

\[ d \sin(\theta + \gamma) + r \cos(\theta) = r + h. \]
Equation 6:

Equation 6 is analogous to Equation 4 in the same way.

\[ d \cos(\theta + \gamma) = r \sin(\theta). \]
Six Equations in Six Unknowns:

\[ \varphi + \beta = \theta + \gamma \]  
\[ \alpha + \beta = 2\gamma \]  
\[ D \sin(\varphi) + r \cos(\varphi) = r + h \]  
\[ D \cos(\varphi) = r \sin(\varphi) \]  
\[ d \sin(\theta + \gamma) + r \cos(\theta) = r + h \]  
\[ d \cos(\theta + \gamma) = r \sin(\theta). \]

Not hard to solve.

Use (2) to solve for \( \gamma \).

Solve (4) for \( D \) and then substitute in for \( D \) in (3).

Solve (6) for \( d \) and then substitute in for \( d \) in (5).

And so on...
Three Equations in Three Unknowns:

\[ \gamma = (\alpha + \beta)/2 \quad (2) \]
\[ D = r \sin(\varphi)/\cos(\varphi) \quad (4) \]
\[ d = r \sin(\theta)/\cos(\theta + \gamma) \quad (6) \]

\[ \varphi - \theta = (\alpha - \beta)/2 \quad (1) \]
\[ r = (r + h) \cos(\varphi) \quad (3) \]
\[ r \cos(\gamma) = (r + h) \cos(\theta + \gamma) \quad (5) \]

Divide (3) and (5) by \( r + h \) and eliminate \( r \):
\[ \cos(\varphi) = \cos(\theta + \gamma)/\cos(\gamma) = \cos(\varphi + \beta)/\cos(\gamma) \]

Expand the cosine of the sum, replace \( \sin(\varphi) \) with \( \sqrt{1 - \cos^2(\varphi)} \) and solve for \( \cos(\varphi) \):
\[ \cos(\varphi) = \sin(\beta)/\sqrt{1 - 2 \cos(\beta) \cos(\gamma) + \cos^2(\gamma)} \]

Substitute this formula for \( \cos(\varphi) \) into (3) and solve for \( r \)...
Answer for \( r \) (radius of Earth) is:

\[
r = \frac{h}{\sqrt{1 - 2 \cos \beta \cos \gamma + \cos^2 \gamma}} - 1
\]

where

\[
\gamma = \frac{\alpha + \beta}{2}.
\]

Plugging in our values for \( \alpha \), \( \beta \), and \( h \), we get

\[r = 4,977 \text{ miles.}\]

Recall that the right answer is 3,960 miles.

Fixing \( \alpha \) and \( \beta \), the height \( h \) that corresponds to this answer is:

\[h = 7 \times \frac{3960}{4977} = 5.56 \text{ feet} = 5' 7''.
\]
Fix $\alpha$. How does the ratio $\beta/\alpha$ vary with $r$...
In terms of pixels...
Morals:

- Always be mindful of units.
- Always draw a picture and label things.
- If there are six unknowns, you need six (distinct) equations.
- A picture need not be to scale; it can exaggerate angles, distances, etc.
Conclusion: ALGEBRA AND GEOMETRY ARE BOTH FUN AND USEFUL.