Hang Glider: Range Maximization

The problem is to compute the flight inputs to a hang glider so as to provide a maximum range flight. This problem first appears in [1].

The hang glider has weight \( W \) (glider plus pilot), a lift force \( L \) acting perpendicular to its velocity \( v_r \) relative to the air, and a drag force \( D \) acting in a direction opposite to \( v_r \). Denote by \( x \) the horizontal position of the glider, by \( v_x \) the horizontal component of the absolute velocity, by \( y \) the vertical position, and by \( v_y \) the vertical component of absolute velocity.

The airmass is not static: there is a thermal just 250 meters ahead. The profile of the thermal is given by the following upward wind velocity:

\[
    u_a(x) = u_m e^{-\left(\frac{x}{R} - 2.5\right)^2 \left(1 - \left(\frac{x}{R} - 2.5\right)^2\right)}.
\]

We take \( R = 100 \) m and \( u_m = 2.5 \) m/s. Note, MKS units are used throughout. The upwind profile is shown in Figure 1.

Letting \( \eta \) denote the angle between \( v_r \) and the horizontal plane, we have the following equations of motion:

\[
    \begin{align*}
    \dot{x} &= v_x, & \dot{v}_x &= \frac{1}{m}(-L \sin \eta - D \cos \eta), \\
    \dot{y} &= v_y, & \dot{v}_y &= \frac{1}{m}(L \cos \eta - D \sin \eta - W)
    \end{align*}
\]

with

\[
    \eta = \arctan \left( \frac{v_y - u_a(x)}{v_x} \right), \quad v_r = \sqrt{v_x^2 + (v_y - u_a(x))^2},
\]

\[
    L = \frac{1}{2} c_L \rho S v_r^2, \quad D = \frac{1}{2} c_D(c_L) \rho S v_r^2, \quad W = mg.
\]

The glider is controlled by the lift coefficient \( c_L \). The drag coefficient is assumed to depend on the lift coefficient as

\[
    c_D(c_L) = c_0 + k c_L^2
\]

where \( c_0 = 0.034 \) and \( k = 0.069662 \). In addition, there is an upper limit on the lift coefficient:

\[
    c_L \leq c_{L,\text{max}} := 1.4.
\]

Other constants are:

\[
    \begin{align*}
    m &= 100 & \text{mass of glider and pilot} \\
    S &= 14 & \text{wing area} \\
    \rho &= 1.13 & \text{air density} \\
    g &= 9.81 & \text{acc due to gravity.}
    \end{align*}
\]
The boundary conditions are:

\[
\begin{align*}
x(0) &= 0, \\
y(0) &= 1000, \\
v_x(0) &= 13.23, \\
v_y(0) &= -1.288, \\
y(T) &= 900, \\
v_x(T) &= 13.23, \\
v_y(T) &= -1.288.
\end{align*}
\]

The total time \( T \) for the flight is, of course, a variable. The objective is to maximize \( x(T) \).

The optimal solution depicted in Figures 2–7 was obtained using a uniform discretization of the time domain into 150 discrete points. Derivatives were approximated by differences at the midpoints of each discrete time interval. The optimal range is 1248.26 m and takes 98.4665 s to fly.

**References**

FIGURE 2. $y$ vs $x$

FIGURE 3. $x$ vs $t$

FIGURE 4. $v_x$ vs $t$

FIGURE 5. $cL$ vs $t$

FIGURE 6. $y$ vs $t$

FIGURE 7. $v_y$ vs $t$