A Martingale System Theorem for Stock Investments

Robert J. Vanderbei

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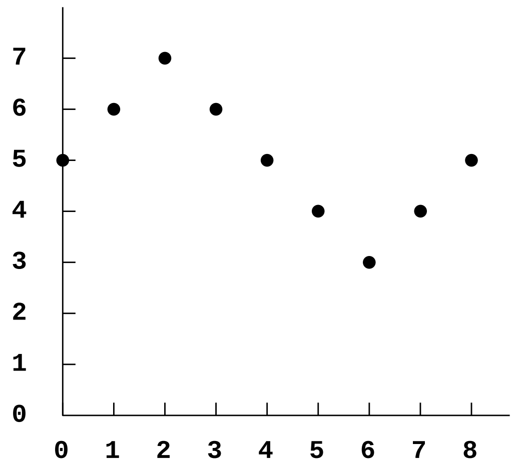
The Beginning

Dollar Cost Averaging (DCA)

Invest a fixed dollar amount each month.
End up buying relatively more when price is low than when high.
Therefore, make money even if market is flat.

Example. Invest $1260 each month.



month	$ / share	shares
0	5	252
1	6	210
2	7	180
3	6	210
4	5	252
5	4	315
6	3	420
7	4	215
		2154

Sell in month 8 @ $5/share	=	$10,770
Total invested (7 x $1260)	=	8,820
Gain	=	$1,950

DCA Theorem

a = $'s invested each month

X_k = share price in month k

$$Y_n = X_n \sum_{k=0}^{n-1} \frac{a}{X_k} - an$$ = net gain up to month n

Theorem. *If X_k, $k = 0, 1, \ldots$, is a simple random walk conditioned to return to its starting point x_0 at time n (and $n < 2x_0$), then $EY_n > 0$.*

Proof. Follows from the reflection principle and the harmonic/arithmetic mean inequality.

Comment

Conditioning the process to return to its starting point at the end is unrealistic.

If we knew that, then we should buy only when the price is lower than the initial price.

Such a strategy would dominate DCA.

The Middle

A Martingale Framework

$\mathcal{F}_k$, $k = 0, 1, \ldots$, a filtration on $(\Omega, \mathcal{F}, P)$

X_k, $k = 0, 1, \ldots$, a positive martingale wrt $\mathcal{F}_k$

A_k, $k = 0, 1, \ldots$, a nonnegative process adapted to $\mathcal{F}_k$

$Y_n = \sum_{k=0}^{n-1} A_k \left(\frac{X_n}{X_k} - 1 \right)$ = net gain up to month n from investment stream A_k.

Theorem. Y_n, $n = 0, 1, \ldots$, *is a mean-zero martingale.*

Proof. Note that

$$Y_{n+1} - Y_n = \sum_{k=0}^{n} \frac{A_k}{X_k} (X_{n+1} - X_n)$$

Take conditional expectations and note adaptedness of A_k / X_k:

$$\mathrm{E}\left[Y_{n+1} - Y_n | \mathcal{F}_n\right] = \sum_{k=0}^{n} \frac{A_k}{X_k} \mathrm{E}\left[X_{n+1} - X_n | \mathcal{F}_n\right] = 0$$

Quadratic Variation

The quadratic variation of X is

$$\langle X \rangle_k = \sum_{j=0}^{k-1} \left(\mathrm{E}\left[X_{j+1}^2 | \mathcal{F}_j \right] - X_j^2 \right)$$

Theorem. *The quadratic variation of the gain process, Y, is given by*

$$\langle Y \rangle_n = \sum_{j=0}^{n-1} \left(\langle X \rangle_{j+1} - \langle X \rangle_j \right) S_j^2,$$

where

$$S_j = \sum_{k=0}^{j} \frac{A_k}{X_k}.$$

Quadratic Variation Proof

Proof. From the previous proof and the definition of process S, we have

$$Y_{n+1} - Y_n = (X_{n+1} - X_n)\, S_n.$$

Since the stock price and gain processes, X and Y, are martingales, we see that

$$\begin{aligned} \mathrm{E}\left[Y_{n+1}^2 | \mathcal{F}_n\right] - Y_n^2 &= \mathrm{E}\left[(Y_{n+1} - Y_n)^2 | \mathcal{F}_n\right] \\ &= \mathrm{E}\left[(X_{n+1} - X_n)^2 | \mathcal{F}_n\right] S_n^2 \\ &= \left(\mathrm{E}\left[X_{n+1}^2 | \mathcal{F}_n\right] - X_n^2\right) S_n^2 \end{aligned}$$

From the definition of quadratic variation, we see that

$$\langle Y \rangle_{n+1} - \langle Y \rangle_n = (\langle X \rangle_{n+1} - \langle X \rangle_n)\, S_n^2.$$

Summing both sides completes the proof.

A Stochastic Calculus Approach

A fundamental result in stochastic calculus is that the stochastic integral of a predictable process against a martingale is again a martingale.

The discrete time analogue is as follows:

Theorem. *If X is a martingale and B is an adapted process, then*

$$Z_n = \sum_{k=0}^{n-1} B_k \left(X_{k+1} - X_k\right)$$

is a martingale.

The fact that the gain process Y is a martingale follows from this theorem by taking

$$B_k = \sum_{j=0}^{k} \frac{A_j}{X_j}.$$

Discounting Trends

Suppose now that X is an arbitrary positive adapted stochastic process.
Put

$$\tilde{X}_k = R_k X_k,$$

where

$$R_k = \prod_{j=0}^{k-1} \frac{X_j}{\mathrm{E}\left[X_{j+1} | \mathcal{F}_j\right]}.$$

The process R is predictable:

$$R_k \in \mathcal{F}_{k-1}, \qquad k = 0, 1, \ldots .$$

Theorem. *The process $\tilde{X}$ is a martingale.*

Proof.

$$\begin{aligned} \mathrm{E}\left[\tilde{X}_{k+1} | \mathcal{F}_k\right] &= R_{k+1} \mathrm{E}\left[X_{k+1} | \mathcal{F}_k\right] \\ &= R_k X_k \\ &= \tilde{X}_k. \end{aligned}$$

Following the Trend

Convert all dollar amounts to discounted (time 0) values.

The gain process is measured in discounted terms:

$$\begin{aligned} Y_n &= R_n X_n S_n - \sum_{k=0}^{n} R_k A_k \\ &= \sum_{k=0}^{n} \left(R_n X_n \frac{A_k}{X_k} - R_k A_k \right) \\ &= \sum_{k=0}^{n} \tilde{A}_k \left(\frac{\tilde{X}_n}{\tilde{X}_k} - 1 \right), \end{aligned}$$

where

$$\tilde{A}_k = R_k A_k$$

represents the initial value of the input at time k.

Our earlier theorem now says that DCA simply tracks the trend in the stock price.

The End

A Continuous Version

X_t = positive continuous local martingale—the share price
B_t = semi-martingale—total amount invested up to time t
$S_t = \int_0^t \frac{dB_u}{X_u}$ = number of shares owned at time t
$Y_t = X_t S_t - B_t$ = net gain at time t

Theorem. *The gain Y_t at time t is given by*

$$Y_t = \int_0^t S_u \, dX_u + \langle X, S \rangle_t .$$

If B is locally of bounded variation, then Y is a continuous local martingale.

Proof

Proof. Start with the stochastic integration by parts formula:

$$X_t S_t = \int_0^t S_u dX_u + \int_0^t X_u dS_u + \langle X, S \rangle_t.$$

From the definition of S, we see that

$$\int_0^t X_u dS_u = B_t.$$

Hence the formula for Y.

If B locally has bounded variation, the S does too.
Hence, the cross variation $\langle X, S \rangle_t$ vanishes.
Since S is continuous, it is predictable and so the remaining stochastic integral is a martingale.

Remarks

1. If shares are purchased but not sold, then B is increasing and hence is locally of bounded variation.

2. If B has unbounded variation, then $\langle X, S \rangle$ is not expected to vanish. In this case, the gain process can be a submartingale—and that's a good thing. See example on next slide.

An Example

Suppose that $B_t = X_t$ for all t.

Using Ito's formula we get

$$S_t = \int_0^t \frac{dB_u}{X_u} = \log(X_t) - \log(X_0) + \frac{1}{2}\int_0^t \frac{d\langle X\rangle_u}{X_u^2}.$$

Hence,

$$\langle X, S\rangle_t = \langle X, \log(X)\rangle_t.$$

Since $\langle X, \log(X)\rangle$ is increasing, the gain process Y is a submartingale.

Conclusion

A seemingly harmless assumption,

the ability to buy and sell very fast with no transaction cost,

leads to a plainly absurd result.

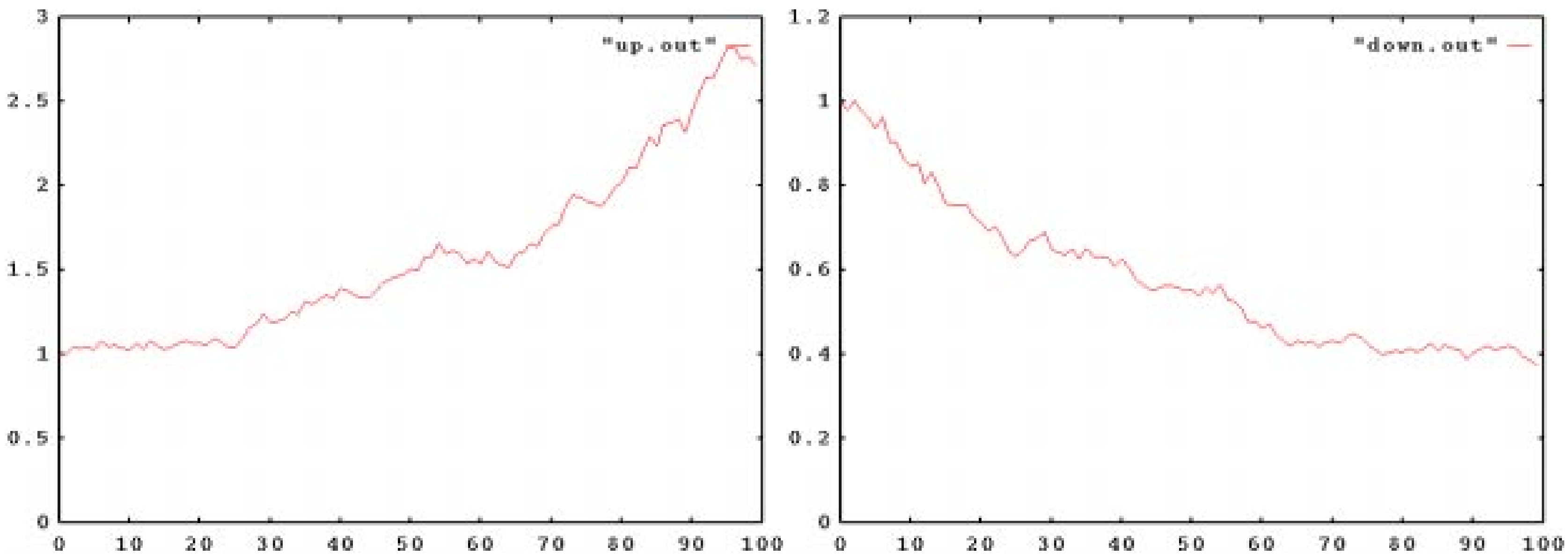
Final Remarks

Other Erroneous Results

The Black-Scholes option pricing formula assumes the ability to trade fast without transaction costs.

The result is a formula that depends on volatility but not on drift.

Consider historical data for companies A (on the left) and B (on the right):



They both have the same volatility. Would you pay the same price for an option on these two companies? (Be honest!)

An Alternative