Using LOQO to Solve Second-Order Cone Programming Problems

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Sixth SIAM Conference on Optimization

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Second Order Cone Programming

Second Order Cone Programming (SOCP) Problem:

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\| \leq c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

Second Order Cone:

$$\{(u, t) \in R^{k+1} : \|u\| \leq t\}$$

The connection:

$$\begin{aligned} u &= A_i x + b_i \\ t &= c_i^T x + d_i \end{aligned}$$

Ref: Lobo, Vandenberghe, Boyd, and Lebret, Applications of second-order cone programming, LAA, to appear.

LOQO

Solves general constrained optimization problems:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h_i(x) \leq 0, \quad i = 1, \ldots, m, \end{array}$$

f and the h_i's are assumed C^2.

Problem is convex if f and the h_i's are all convex.

For SOCP,

$$h_i(x) = \|A_i x + b_i\| - c_i^T x - d_i$$

is convex

but not differentiable on

$$\{x : A_i x + b_i = 0\}.$$

Ref: Vanderbei and Shanno, An Interior-Point Algorithm for Nonconvex Nonlinear Programming, COAP, to appear.

Nondifferentiability

Nondifferentiability is probably not be a problem unless it happens at optimality.

Example:

$$\begin{array}{ll} \text{minimize} & ax_1 + x_2 \\ \text{subject to} & |x_1| \le x_2, \end{array}$$

where

$$-1 < a < 1.$$

Clearly, $(x_1^*, x_2^*) = (0, 0)$.

Dual feasibility:

$$\begin{bmatrix} a \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{d|x_1|}{dx_1} \\ -1 \end{bmatrix} y = 0.$$

LOQO must pick the correct value for $\frac{d|x_1|}{dx_1}$ when $x_1 = 0$:

$$\left.\frac{d|x_1|}{dx_1}\right|_{x_1=0} = -a.$$

Not possible a priori.

Alternate Formulations

Constraint formulation:

$$\phi(A_i x + b_i, c_i^T x + d_i) \leq 0, \quad i = 1, \ldots, m$$

where

$$\phi(u, t) = \|u\| - t.$$

Not differentiable at $u = 0$.

Smooth alternatives:

$\sqrt{\epsilon^2 + \sum_i u_i^2} - t \leq 0,$	convex	**not equiv.**	
$\sqrt{v^2 + \sum_i u_i^2} - t \leq 0, \;\; v \geq \epsilon$	convex	**not equiv.**	
$\|u\|^2 - t^2 \leq 0, \;\; t \geq 0$	**nonconvex**	equiv.	
$\frac{\|u\|^2}{t} - t \leq 0, \;\; t > 0$	convex	equiv.	**strict interior**
$e^{(\|u\|^2 - t^2)/2} - 1 \leq 0, \;\; t \geq 0$	convex	equiv.	**badly scaled**

Applications

Antenna array weight design.

Finite impulse response (FIR) filters.

Structural optimization.

Euclidean multifacility location.

Minimal Surfaces.

Markowitz model for portfolio optimization.

Grasping force minimization.

Springs in equilibrium.

See: http://www.sor.princeton.edu/∼rvdb/ampl/nlmodels

Antenna Array Weight Design

$$\begin{aligned} \text{minimize} \quad & \rho \\ \text{subject to} \quad & |A(p)|^2 \le \rho, \qquad p \in S \\ & A(p_0) = 1, \end{aligned}$$

where

$$\begin{aligned} A(p) &= \sum_{l \in \{\text{array elements}\}} w_l e^{-2\pi i p \cdot l}, \qquad p \in S \\ w_l &= \text{complex-valued } \textbf{design weight} \text{ for array element } l \\ S &= \text{subset of unit hemisphere: sidelobe directions} \\ p_0 &= \text{“look” direction} \end{aligned}$$

Classification: convex, QLP, not SOCP.

Specific Example: Hexagonal Lattice of 61 Elements

$$\begin{aligned} \rho &= -20 \text{ dB} = 0.01 \\ S &= 889 \text{ points outside } 20^\circ \text{ from look direction} \\ p_0 &= 40^\circ \text{ from zenith} \end{aligned}$$

constraints	839
variables	123
time (secs)	
LOQO	722
MINOS	>60000
LANCELOT	55462
SNOPT	—

Antenna Array Pattern Synthesis as a Convex Optimization Problem

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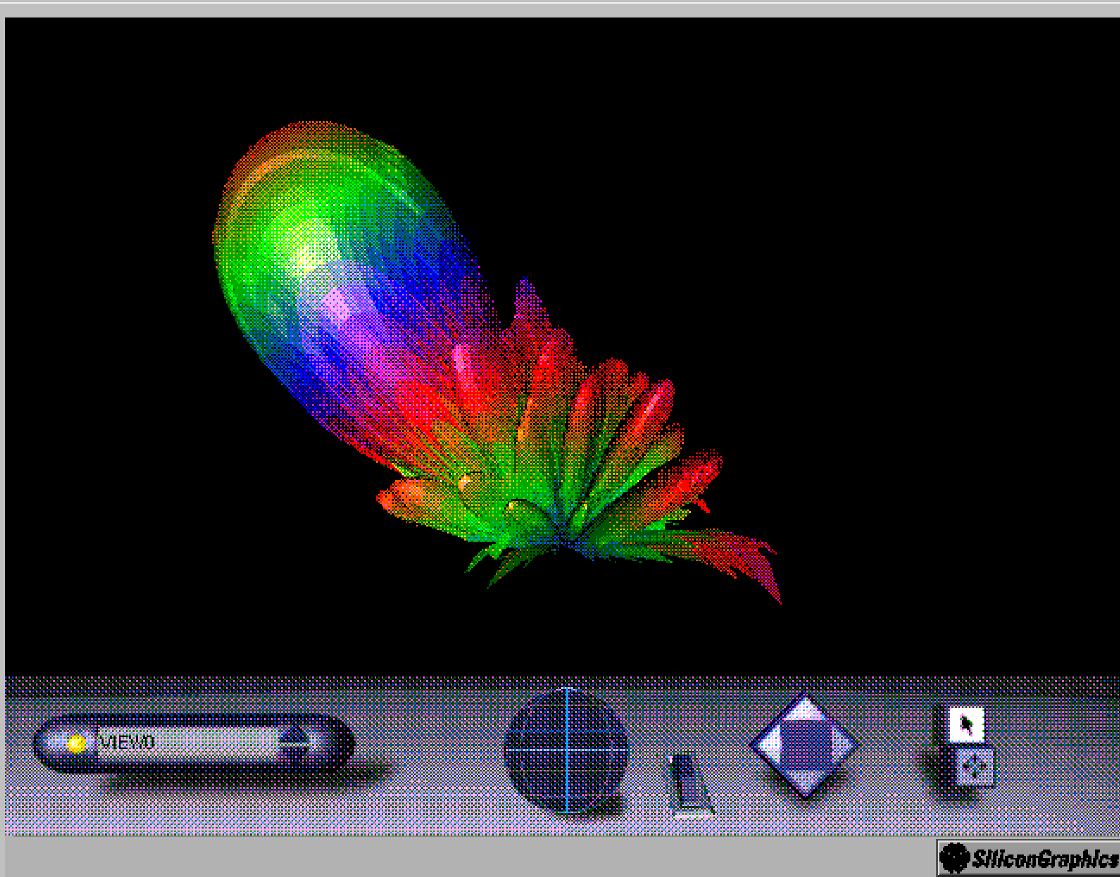


Given an array of antennae with a given specific layout, compute the complex coefficients with which to multiply each individual signal before adding them together so that the combined signal has unit gain in a specific direction, the so–called *look direction*, and the smallest average gain possible outside a certain neighborhood of the look direction.

The specific problem whose solution is shown above involved a hexagonally shaped hexagonal lattice, a look direction that is 40 degrees off *bore sight* (which is straight up), and the requirement to minimize the response at all angles 20 degrees or more away from the look direction. The large lobe is aligned along the look direction. The plot is in decibels vs direction. The peak of the main lobe corresponds to 0 dB. Each color band (green, blue, red, etc.) corresponds to a 10 dB reduction in signal strength. To see the AMPL file that was used to generate the surface shown above, click here.

Finite Impulse Response (FIR) Filter Design

$$\begin{aligned}
&\text{minimize} \quad \rho \\
&\text{subject to} \quad \int_0^1 (H_w(\nu) + H_m(\nu) + H_t(\nu) - 1)^2 \, d\nu \le \epsilon \\
&\qquad\qquad \int_W H_w^2(\nu) d\nu \le |W|\rho \qquad W = [.2, .8] \\
&\qquad\qquad \int_M H_m^2(\nu) d\nu \le |M|\rho \qquad M = [.4, .6] \cup [.9, .1] \\
&\qquad\qquad \int_T H_t^2(\nu) d\nu \le |T|\rho \qquad T = [.7, .3]
\end{aligned}$$

where

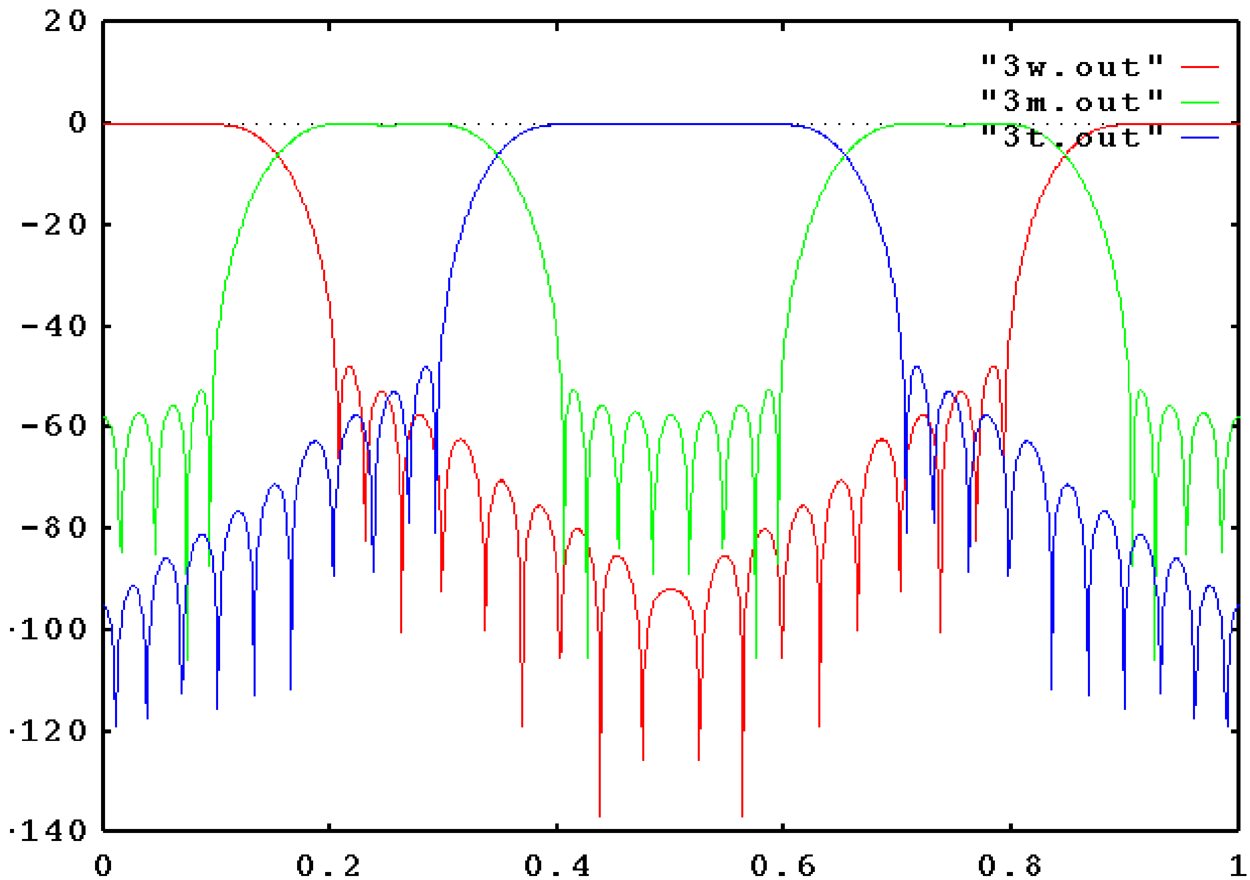
$$\begin{aligned}
H_i(\nu) &= h_i(0) + 2\sum_{k=1}^{n-1} h_i(k) \cos(2\pi k \nu), \quad i = w, m, t \\
h_i(k) &= \text{filter coefficients, i.e., } \textbf{decision variables}
\end{aligned}$$

Classification: convex, QLP, not SOCP.

Specific Example: Pink Floyd

filter length: $n = 14$ integral discretization: $N = 1000$

constraints	4
variables	43
time (secs)	
LOQO	79
MINOS	164
LANCELOT	3401
SNOPT	35



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"3m.out"
"3t.out"
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0
-20
-40
-60
-80
-100
-120
-140
0
0.2
0.4
0.6
0.8
1

Structural Design

$$\text{minimize} \quad -p^T w$$

$$\text{subject to} \quad \frac{V}{A_e} w^T K_e w \leq 1, \quad e \in \mathcal{E}$$

where

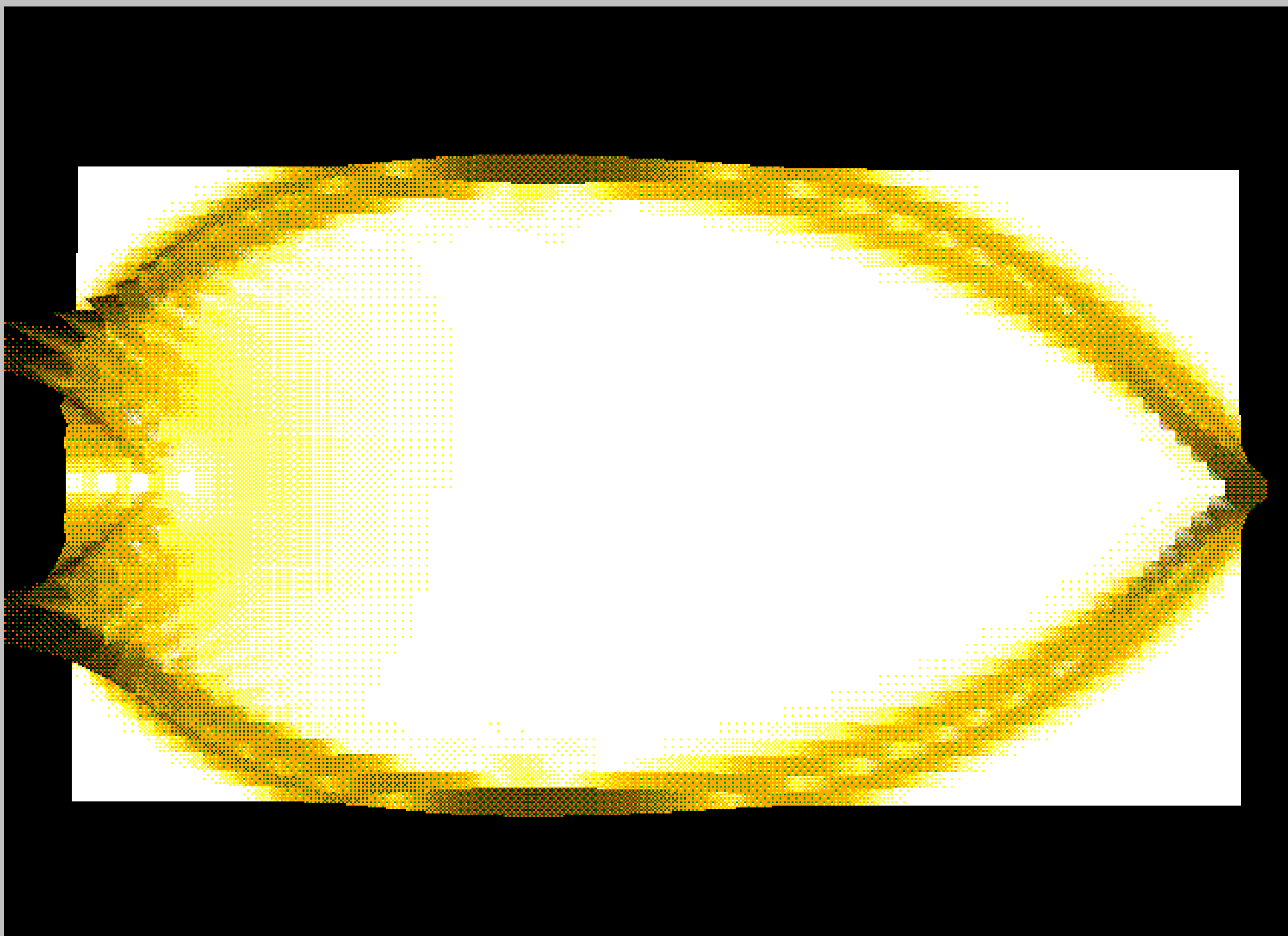
$$\begin{aligned} p &= \text{applied load} \\ w &= \text{node displacements; } \textbf{optimization vars} \\ V &= \text{total volume} \\ A_e &= \text{thickness of element } e \\ K_e &= \text{element stiffness matrix } (\succeq 0) \\ \mathcal{E} &= \text{set of elements} \end{aligned}$$

Classification: convex, QLP, not SOCP.

Specific Example: Michel Bracket

element grid	40x72	20x36	5x9
constraints	2880	720	45
variables	5965	1536	112
time (secs)			
LOQO	412	89.7	2.32
MINOS	∞	(IL)	(BS)
LANCELOT	∞	∞	15.73
SNOPT	-	(IS)	(BS)

Minimal Compliance Bracket as a Convex Optimization Problem



Given some points in space at which a bracket is to be anchored, to a wall say, and other points at which a load is to be supported by the bracket, the minimal compliance bracket design problem is to design a bracket from a given total amount of material that can support the given load and that has minimum compliance (which is the same thing as maximum stiffness).

Minimal Surfaces

$$\text{minimize} \quad \iint_D \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} dxdy$$

$$\text{subject to} \quad u(x, y) = \gamma(x, y), \quad (x, y) \in \partial D.$$

where

$$\begin{aligned} D &= \text{domain in } xy\text{-plane} \\ u(x, y) &= \text{height of surface above } xy\text{-plane, i.e., } \textbf{optimization variables} \\ \gamma(x, y) &= \text{boundary data} \end{aligned}$$

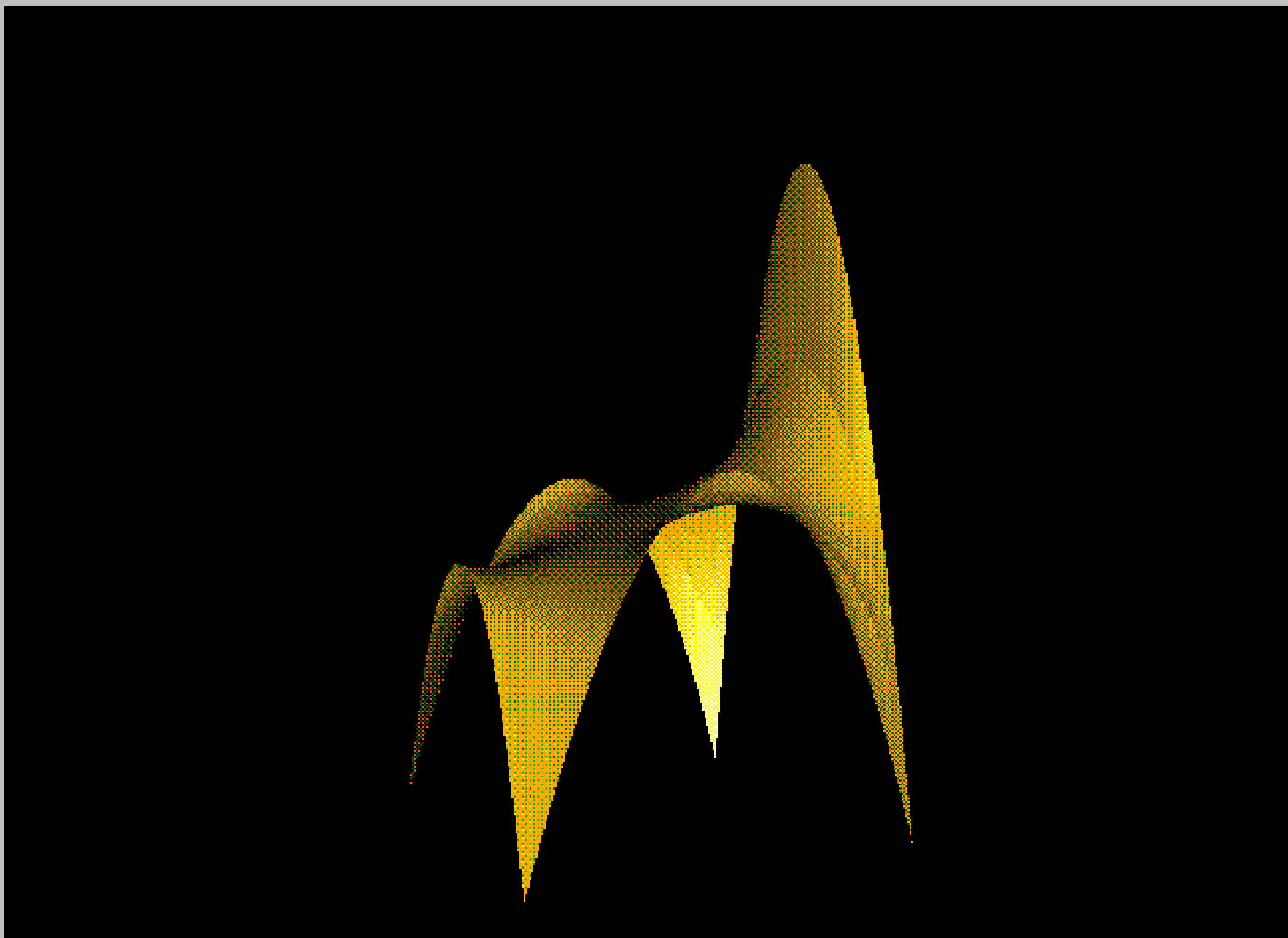
Classification: smooth, convex, not SOCP.

Specific Example: A "Chair"

$$\begin{aligned} D &= \text{Square} \\ \gamma &= \text{concave parabolas on each side} \end{aligned}$$

discrete grid	41x41
constraints	172
variables	1849
time (secs)	
LOQO	13
MINOS	3652
LANCELOT	100
SNOPT	2963

Minimal Surface as a Convex Optimization Problem



Given a domain in R^2 and boundary data, the minimal surface problem is to find an interpolation of the boundary data into the interior of the domain so that the surface so generated has minimal surface area. This problem is a convex optimization problem. To see the AMPL file that was used to generate the surface shown above, click here.

Multifacility Location

$$\text{minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \|x_j - a_i\| + \sum_{j=1}^{n} \sum_{j'=1}^{j-1} v_{jj'} \|x_j - x_{j'}\|.$$

where

$$\begin{aligned} a_i &= \text{location of existing facilities, } i = 1, \dots, m \\ x_j &= \text{location of new facilities, } j = 1, \dots, n \end{aligned}$$

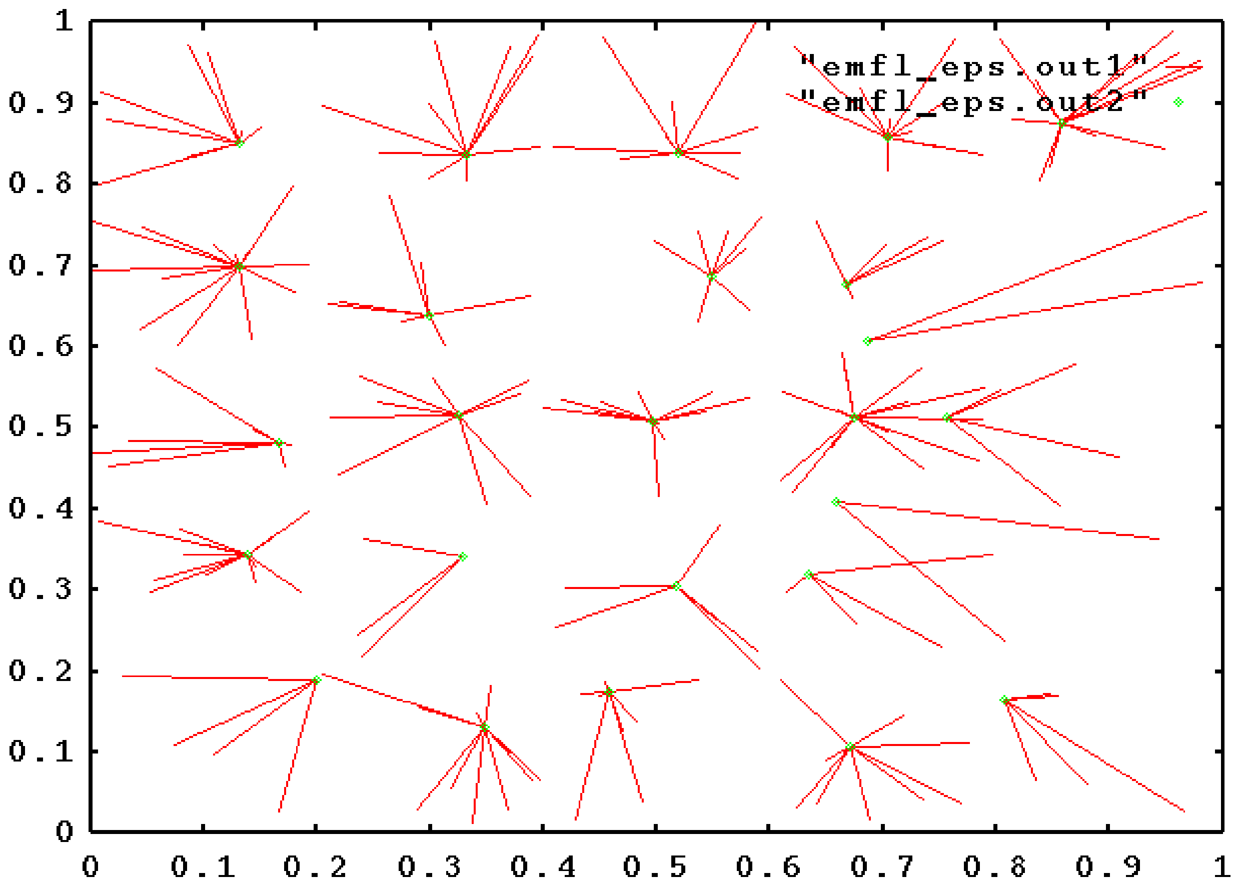
Classification: not smooth, convex, not SOCP.

Specific Example: Randomly Generated

$$\begin{aligned} m &= 200 \\ n &= 25 \end{aligned}$$

Used v-perturbation for smoothing.

constraints	0
variables	1849
time (secs)	
LOQO	2.3
MINOS	9.7
LANCELOT	11.0
SNOPT	4.7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"emfl_eps.out2"
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0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Comparisons Among Alternative Formulations and Solvers

Problem	m	n	SOCP?	LOQO	MINOS	SNOPT	LANCELOT
antenna	166	49		21.36	*	*	*
antenna_socp	167	49	Y	16.65	*	*	*
antenna_ratio	167	49	Y	*	*	*	*
antenna_nonconvex	166	49	Y	*	*	*	*
antenna_exp	166	49	Y	57.85	*	*	*
fir_convex	243	11		0.99	1.38	1.52	11.58
fir_socp	244	12	Y	0.89	0.96	1.04	9.27
fir_ratio	244	12	Y	1.00	1.08	1.08	11.05
structure2 (LP)	1910	176		4.04	21.39	23.09	497.75
structure3 (convex QLP)	960	176		39.87	29.54	125.45	3043.75
structure_socp	1145	2886	Y	127.71	*	*	*
structure_socp_eps	1145	2886	Y	105.11	*	*	*
structure_ratio	1145	2886	Y	122.51	*	*	*
structure_nonconvex	1145	2886	Y	*	*	*	*
structure_exp	1145	2886	Y	*	*	*	*
minsurf.mod	0	961		4.91	688.95	*	32.33
minsurf_socp.mod	2046	3009	Y	90.96	5634.04	*	*
minsurf_nonconvex2.mod	2048	3009	Y	731.75	*	*	*
minsurf_exp.mod	2048	3009	Y	*	*	*	*
minsurf_ratio.mod	2048	3009	Y	107.7	6856.60	*	*
emfl.mod	0	50		*	*	*	*
emfl_eps.mod	0	50		2.09	5.99	1.71	3.80
emfl_socp.mod	5300	5350	Y	*	*	*	*
emfl_socp_eps.mod	5300	5350	Y	753.01	*	3368.51	*
emfl_nonconvex.mod	5300	5350	Y	*	*	*	*
emfl_exp.mod	5300	5350	Y	*	*	*	*
emfl_ratio.mod	5300	5350	Y	868.79	*	*	*

Comparison with an SOCP Code

Comparison between:

- LOQO called from AMPL
- LOQO called from MATLAB
- Lobo, Vandenberghe, and Boyd's code called from MATLAB

	AMPL			MATLAB			
Problem	*m*	*n*	LOQO	*m*	*n*	LOQO	LVB
antenna_socp	167	49	4.21	187	59	9.03	**
fir_socp	243	5	0.11	307	5	0.12	1.58
structure_socp	155	336	0.67	442	357	2.35	*
emfl_socp_eps	86	94	0.12	172	94	0.12	18.62
minsurf_socp	70	97	0.07	100	149	0.18	68.83

Comparison with an SDP Code

Comparison between:

- LOQO called from MATLAB
- Jos Sturm's SEDUMI called from MATLAB

	m	*n*	SigFigs	LOQO	SEDUMI
antenna_socp	177	49	6	28.6	*
fir_socp	307	5	8	0.12	0.39
structure_socp	421	336	8	1.08	4.54
emfl_socp_eps	86	94	11	0.13	0.51
minsurf_socp	70	97	10	0.08	0.60