Diffraction Analysis of 2-D Pupil Remapping for High-Contrast Imaging

Robert J. Vanderbei

SPIE San Diego
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Princeton University
http://www.princeton.edu/~rvdb
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1. Apodization

![Diagram of apodization effect](image)
\begin{align*}
E(\xi, \eta) &= \frac{1}{\lambda f} \int \int e^{2\pi i \frac{\tilde{x}\xi + \tilde{y}\eta}{\lambda f}} A(\sqrt{\tilde{x}^2 + \tilde{y}^2}) d\tilde{y} d\tilde{x} \\
E(\rho) &= \frac{2\pi}{\lambda f} \int J_0 \left( 2\pi \frac{\tilde{r} \rho}{\lambda f} \right) A(\tilde{r}) \tilde{r} d\tilde{r} .
\end{align*}

\text{Psf}(\rho) = |E(\rho)|^2
Pupil Mapping

Advantages:

- 100% throughput
- Implicit magnification... effectively $iwa \approx 1\lambda/D$.

Disadvantages:

- Diffraction effects limit achievable contrast to $10^{-5}$ for a pure pupil-mapping system.
3. Notations

\[ R(\tilde{r}) = \pm \sqrt{\int_{0}^{\tilde{r}} 2A^2(s)ds}. \]

\[ \frac{\partial h}{\partial r}(r) = \frac{r - \tilde{R}(r)}{\sqrt{Q_0^2 + (n^2 - 1)(r - \tilde{R}(r))^2}} \]

\[ \frac{\partial \tilde{h}}{\partial \tilde{r}}(\tilde{r}) = \frac{R(\tilde{r}) - \tilde{r}}{\sqrt{Q_0^2 + (n^2 - 1)(R(\tilde{r}) - \tilde{r})^2}} \]

\( n \) is the refractive index and

\[ Q_0 = -(n - 1)z \]
\[ = S(R(\tilde{r}), \tilde{r}) + n(\tilde{h}(\tilde{r}) - h(R(\tilde{r}))) \]

Notes:

- the second expression for \( Q_0 \) is in fact independent of \( \tilde{r} \).
- for mirrors, put \( n = -1 \).
Huygens Wavelets

Cartesian Coordinates

\[ E_{\text{out}}(\tilde{x}, \tilde{y}) = \int \int \frac{1}{\lambda Q(\tilde{x}, \tilde{y}, x, y)} e^{2\pi i Q(\tilde{x}, \tilde{y}, x, y)/\lambda} dy dx, \]

where \( Q \) denotes the optical path length:

\[ Q(\tilde{x}, \tilde{y}, x, y) = \sqrt{(x - \tilde{x})^2 + (y - \tilde{y})^2 + (h(r) - \tilde{h}(\tilde{r}))^2 + n(Z - h(r) + \tilde{h}(\tilde{r}))} \]

Polar Coordinates

\[ E_{\text{out}}(\tilde{r}) = \int \int \frac{1}{\lambda Q(\tilde{r}, r, \theta)} e^{2\pi i Q(\tilde{r}, r, \theta)/\lambda} r d\theta dr, \]

where

\[ Q(\tilde{r}, r, \theta) = \sqrt{r^2 - 2r\tilde{r} \cos \theta + \tilde{r}^2 + (h(r) - \tilde{h}(\tilde{r}))^2 + n(Z - h(r) + \tilde{h}(\tilde{r}))}. \]

Double integrals are hard!
Fresnel Approximation

Assuming a *large separation* between the lenses and that the lenses are themselves *thin*, we get

\[
\sqrt{r^2 - 2rr\tilde{r} \cos \theta + \tilde{r}^2 + (h(r) - \tilde{h}(\tilde{r}))^2} \\
\approx (h(r) - \tilde{h}(\tilde{r})) + \frac{r^2 - 2rr\tilde{r} \cos \theta + \tilde{r}^2}{2z}.
\]

Fresnel uses this approximation in the exponential (and the “paraxial” approximation in the leading factor) to get the *standard Fresnel approximation*:

\[
E_{\text{out}}(\tilde{r}) = \frac{2\pi}{\lambda Z} e^{\pi i \frac{\tilde{r}^2}{2\lambda} + 2\pi i \frac{(n-1)\tilde{h}(\tilde{r})}{\lambda}} \int e^{\pi i \frac{r^2}{2\lambda} - 2\pi i \frac{(n-1)h(r)}{\lambda}} J_0(2\pi r\tilde{r}/\lambda) r dr.
\]
Fresnel Examples

Flat Glass $A \equiv 1$

$n = 1.5. \ D = 25\text{mm.} \ z = 15D. \ \lambda = 632.8\text{nm.}$
Galilean Telescope: \( A \equiv a \)

\[
R(\tilde{r}) = a\tilde{r} \quad \tilde{R}(r) = \frac{r}{a}
\]

\[
h(r) = z + \frac{\sqrt{Q_0^2 + (n^2 - 1)(1 - 1/a)^2r^2} - |Q_0|}{(n^2 - 1)(1 - 1/a)}
\]

\[
\tilde{h}(\tilde{r}) = \frac{\sqrt{Q_0^2 + (n^2 - 1)(a - 1)^2\tilde{r}^2} - |Q_0|}{(n^2 - 1)(a - 1)}
\]

If \( n > 1 \), then both lenses are hyperbolic.

If \( n < 1 \), then both lenses are elliptical.
Galilean Telescope \( A \equiv 3 \)

Fresnel approximation is very bad.
Better Than Fresnel (BTF)

We retain the paraxial approximation for the leading factor.

We can subtract an arbitrary constant from the $Q$ in the exponent.

Let’s choose to subtract $Q(\tilde{r}, R(\tilde{r}), 0)$. We compute:

$$Q(\tilde{r}, r, \theta) - Q(\tilde{r}, R(\tilde{r}), 0) = S(\tilde{r}, r, \theta) - S(\tilde{r}, R(\tilde{r}), 0) + n(h(R(\tilde{r})) - h(r))$$

$$= \frac{S^2(\tilde{r}, r, \theta) - S^2(\tilde{r}, R(\tilde{r}), 0)}{S(\tilde{r}, r, \theta) + S(\tilde{r}, R(\tilde{r}), 0)} + n(h(R(\tilde{r})) - h(r))$$

Then, we simplify the numerator:

$$S^2(\tilde{r}, r, \theta) - S^2(\tilde{r}, R(\tilde{r}), 0) = (r - R(\tilde{r}))(r + R(\tilde{r})) - 2\tilde{r} \left( r \cos \theta - R(\tilde{r}) \right)$$

$$+ (h(r) - h(R(\tilde{r}))) \left( h(r) + h(R(\tilde{r})) - 2\tilde{h}(\tilde{r}) \right).$$
Mo’ Better

So far, everything is exact (except for the paraxial approximation). The only other approximation is to replace $S(\tilde{r}, r, \theta)$ in the denominator with $S(\tilde{r}, R(\tilde{r}), 0)$ so that the denominator becomes just $2S(\tilde{r}, R(\tilde{r}), 0)$. Replacing the integral on $\theta$ with the appropriate Bessel function, we get a new approximation:

$$E_{out}(\tilde{r}) \approx \frac{2\pi}{\lambda S} \int e^{2\pi i \left( \frac{r^2 - R(\tilde{r})^2 + 2rR(\tilde{r}) + (h(r) - h(R(\tilde{r}))) \left( h(r) + h(R(\tilde{r})) - 2\tilde{h}(\tilde{r}) \right)}{2S(\tilde{r}, R(\tilde{r}), 0)\lambda} + \frac{n}{\lambda} (h(R(\tilde{r})) - h(r)) \right) \cdot J_0 \left( \frac{2\pi \tilde{r}r}{\lambda S(\tilde{r}, R(\tilde{r}), 0)} \right) } r dr.$$
Example: \( A \equiv 3 \)

This approximation is much better than Fresnel.
Pupil Mapping for High Contrast (PIAA)

Designed for $10^{-10}$. Delivers $10^{-5}$. 
Brute Force Huygens Wavelet Integral

Full 2D integration: 500 $r$-values and 500 $\theta$-values. Increased the wavelength by a factor 10. The degradation is fundamental physics, not numerical error.
How about larger lenses?

\[ D = 2.5\text{m} \text{ (100 times bigger). Discretization: 500000 points. First sidelobe: } 1.3 \times 10^{-7}. \]
A 10 inch primary and a 2 inch secondary.

A DISASTER!
Design for Medium Contrast

 Designed for $10^{-5}$. Delivers $10^{-5}$.

 $\max(A)$ is smaller...less off-axis distortion to correct.
Medium Contrast Pupil Mapping with Back-End Apodizer

Hybrid system designed for $10^{-10}$. Delivers $10^{-7.5}$. 
This hybrid system is about as good as the back-end apodizer. Easier to manufacture.