Solving Trajectory Optimization Probems as Large-Scale Nonlinear Programs

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1 Outline

- Celestial Mechanics
- Putting on an Uneven Green
- Goddard Rocket Problem

2 Celestial Mechanics—Periodic Orbits

- Find periodic orbits for the planar gravitational n-body problem.
- Minimize action:

$$\int_0^{2\pi} (K(t) - P(t))dt,$$

- where $K(t)$ is kinetic energy,

$$K(t) = \frac{1}{2} \sum_i \left(\dot{x}_i^2(t) + \dot{y}_i^2(t) \right),$$

- and $P(t)$ is potential energy,

$$P(t) = -\sum_{i<j} \frac{1}{\sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2}}.$$

- Subject to periodicity constraints:

$$x_i(2\pi) = x_i(0), \qquad y_i(2\pi) = y_i(0).$$

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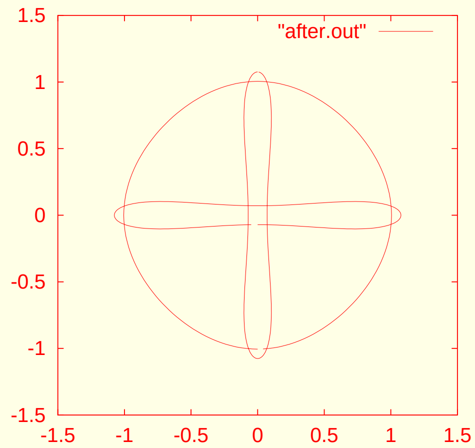
3 Specific Example

Orbits.mod with $n = 3$ and $(0, 2\pi)$ discretized into a 160 pieces gives the following results:

constraints	0	
variables	960	
time (secs)		
LOQO	1.1	
SNOPT	287	(no change for last 80% of iterations)



4 Putting on an Uneven Green

Given:

- $z(x, y)$ elevation of the green.
- Starting position of the ball (x_0, y_0).
- Position of hole (x_f, y_f).
- Coefficient of friction μ.

Find: initial velocity vector so that ball will roll to the hole and arrive with minimal speed.

Variables:

- $u(t) = (x(t), y(t), z(t))$—position as a function of time t.
- $v(t) = (v_x(t), v_y(t), v_z(t))$—velocity.
- $a(t) = (a_x(t), a_y(t), a_z(t))$—acceleration.
- T—time at which ball arrives at hole.

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5 Putting—Two Approaches

- Problem can be formulated with two decision variables:

  $$v_x(0) \quad \text{and} \quad v_y(0)$$

  and two constraints:

  $$x(T) = x_f \quad \text{and} \quad y(T) = y_f.$$

  In this case, $x(T)$, $y(T)$, and the objective function are complicated functions of the two variables that can only be computed by integrating the appropriate differential equation.

- A discretization of the complete trajectory (including position, velocity, and acceleration) can be taken as variables and the physical laws encoded in the differential equation can be written as constraints.

To implement the first approach, one would need an ode integrator that provides, in addition to the quantities being sought, first and possibly second derivatives of those quantities with respect to the decision variables.

We advocate the second approach.

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6 Putting—Continued

Objective:

$$\text{minimize } v_x(T)^2 + v_y(T)^2.$$

Constraints:

$$\begin{aligned} v &= \dot{u} \\ a &= \dot{v} \\ ma &= N + F - mge_z \\ u(0) &= u_0 \qquad u(T) = u_f, \end{aligned}$$

where

- m is the mass of the golf ball.
- g is the acceleration due to gravity.
- e_z is a unit vector in the positive z direction.

and ...

6 Putting—Continued

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and ...

7 Putting—Continued

- $N = (N_x, N_y, N_z)$ is the normal force:

$$\begin{aligned} N_z &= m\frac{g - a_x(t)\frac{\partial z}{\partial x} - a_y(t)\frac{\partial z}{\partial y} + a_z(t)}{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1} \\ N_x &= -\frac{\partial z}{\partial x}N_z \\ N_y &= -\frac{\partial z}{\partial y}N_z. \end{aligned}$$

- F is the force due to friction:

$$F = -\mu\|N\|\frac{v}{\|v\|}.$$

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\end{aligned}
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- F is the force due to friction:

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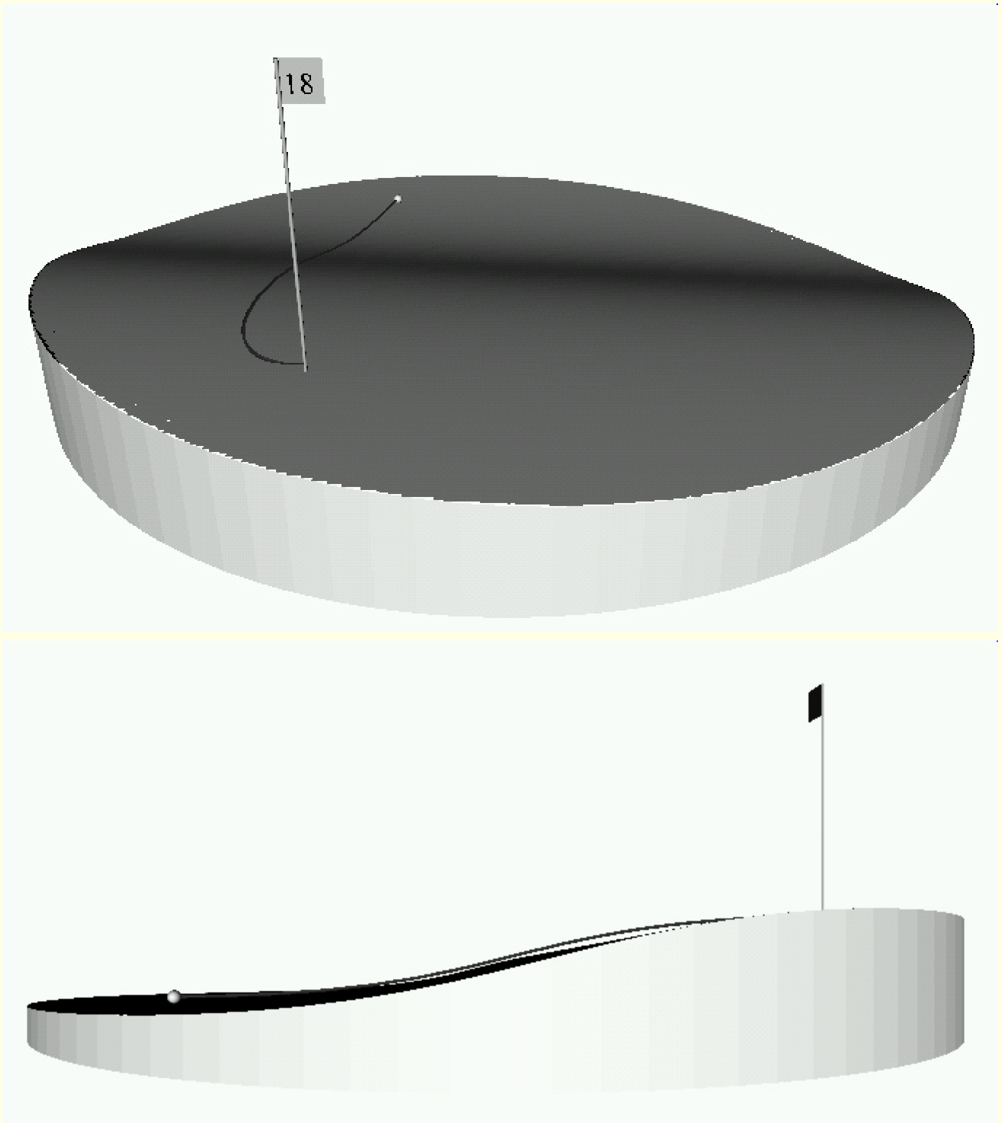
8 Putting—Specific Example

- Discretize continuous time into $n = 200$ discrete time points.
- Use finite differences to approximate the derivatives.



constraints	597
variables	399
time (secs)	
LOQO	14.1
SNOPT	4.1

9 Goddard Rocket Problem

Objective:

$$\text{maximize } h(T);$$

Constraints:

$$\begin{aligned} v &= \dot{h} \\ a &= \dot{v} \\ \theta &= -c\dot{m} \\ ma &= (\theta - \sigma v^2 e^{-h/h_0}) - gm \\ 0 &\leq \theta \leq \theta_{\max} \\ m &\geq m_{\min} \\ h(0) &= 0 \quad v(0) = 0 \quad m(0) = 3 \end{aligned}$$

where

- $\theta =$ Thrust , $m =$ mass
- $\theta_{\max}$, g, σ, c, and h_0 are given constants
- h, v, a, T_h, and m are functions of time $0 \leq t \leq T$.

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Goddard Rocket Problem—Solution

constraints	**399**
variables	**599**
time (secs)	
LOQO	**5.2**
SNOPT	(IL)