Engineering Applications of Nonlinear Optimization

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ABSTRACT

- Brief description of interior-point methods for NLP.
- Engineering Applications
  - Finite Impulse Response (FIR) filter design
  - Antenna Array design
  - Telescope design
  - Stable orbits for the n-body problem

LOQO: An Interior-Point Code for NLP

LOQO solves problems in the following form:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & b \le h(x) \le b + r, \\ & l \le x \le u \end{array}$$

The functions $f(x)$ and $h(x)$ must be twice differentiable (at least at points of evaluation).

The standard *interior-point paradigm* is used:

- Add slacks.
- Replace nonnegativities with barrier terms in objective.
- Write first-order optimality conditions.
- Rewrite optimality conditions in primal-dual symmetric form.
- Use Newton's method to get search directions...

Interior-Point Paradigm Continued

- Use Newton's method to get search directions:

$$\begin{bmatrix} -H(x,y) - D & A^T(x) \\ A(x) & E \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \nabla f(x) - A^T(x) y \\ -h(x) + \mu Y^{-1} e \end{bmatrix}.$$

Here, D and E are diagonal matrices involving slack variables,

$$H(x,y) = \nabla^2 f(x) - \sum_{i=1}^{m} y_i \nabla^2 h_i(x) + \lambda I, \text{ and } A(x) = \nabla h(x),$$

where λ is chosen to ensure appropriate descent properties.

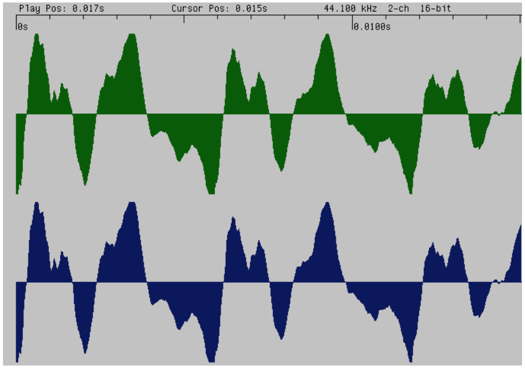
- Compute step lengths to ensure positivity of slack variables.
- Shorten steps further to ensure a reduction in either infeasibility or in the barrier function—a myopic, or Markov, *filter*. (N.B.: We no longer use a merit function.)
- Step to new point and repeat.

Finite Impulse Response (FIR) Filter Design

- Audio is stored digitally in a computer as a stream of short integers: u_k, $k \in \mathbb{Z}$.

- When the music is played, these integers are used to drive the displacement of the speaker from its resting position.

- For CD quality sound, 44100 short integers get played per second per channel.



0	-32768	8	-23681	16	12111
1	-32768	9	-18449	17	17311
2	-32768	10	-11025	18	21311
3	-30753	11	-6913	19	23055
4	-28865	12	-4337	20	23519
5	-29105	13	-1329	21	25247
6	-29201	14	1743	22	27535
7	-26513	15	6223	23	29471

FIR Filter Design—Continued

- A *finite impulse response (FIR) filter* takes as input a digital signal and convolves this signal with a finite set of fixed numbers $h_0, \ldots, h_n$ to produce a filtered output signal:

$$y_k = \sum_{i=-n}^{n} h_{|i|} u_{k-i}.$$

- Sparing the details, the output power at frequency ν is given by

$$|H(\nu)|^2$$

where

$$H(\nu) = \sum_{k=-n}^{n} h_{|k|} e^{2\pi i k \nu} = h(0) + 2 \sum_{k=1}^{n} h_k \cos(2\pi k \nu),$$

- Similarly, the mean absolute deviation from a flat frequency response over a frequency range, say $\mathcal{L} \subset [0,1]$, is given by

$$\frac{1}{|\mathcal{L}|} \int_{\mathcal{L}} |H(\nu) - 1| \, d\nu$$

Filter Design: Woofer, Midrange, Tweeter

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\int_0^1 \left| H_w(\nu) + H_m(\nu) + H_t(\nu) - 1 \right| d\nu \\
\text{subject to} & -\epsilon \le H_w(\nu) \le \epsilon, \qquad \nu \in W = [.2, .8] \\
& -\epsilon \le H_m(\nu) \le \epsilon, \qquad \nu \in M = [.4, .6] \cup [.9, .1] \\
& -\epsilon \le H_t(\nu) \le \epsilon, \qquad \nu \in T = [.7, .3]
\end{array}
$$

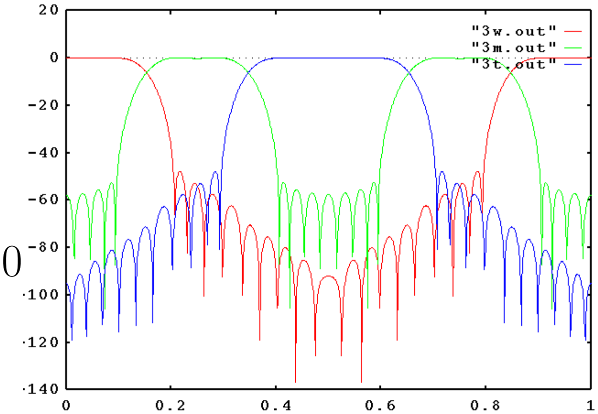
where

$$
\begin{aligned}
H_i(\nu) &= h_0^i + 2 \sum_{k=1}^{n} h_k^i \cos(2\pi k \nu), \quad i = W, M, T \\
h_k^i &= \text{filter coefficients, i.e., } \textbf{decision variables}
\end{aligned}
$$

Conversion to a Linear Programming Problem

$$
\begin{aligned}
\text{minimize} \quad & \int_0^1 t(\nu)d\nu \\
\text{subject to} \quad & t(\nu) \le H_w(\nu) + H_m(\nu) + H_t(\nu) - 1 \le t(\nu) \quad && \nu \in [0,1] \\
& -\epsilon \le H_w(\nu) \le \epsilon, && \nu \in W \\
& -\epsilon \le H_m(\nu) \le \epsilon, && \nu \in M \\
& -\epsilon \le H_t(\nu) \le \epsilon, && \nu \in T
\end{aligned}
$$

Specific Example



$$\text{filter length: } n = 14$$
$$\text{integral discretization: } N = 1000$$

Demo: orig-clip woofer midrange tweeter reassembled

Ref: J.O. Coleman, U.S. Naval Research Laboratory,

CISS98 paper available: engr.umbc.edu/∼jeffc/pubs/abstracts/ciss98.html

Antenna Array Design

- Given: an array (linear or 2-D) of radar antennae.
- An incoming signal generates an output signal at each antenna.
- A linear combination of the signals is made to produce one total output signal.
- Coefficients of the linear combination can be chosen to accentuate and/or attenuate the output signal's strength as a function of the input signal's source direction.

2-D Antenna-Array Design Problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\int_S |A(p)|^2 ds \\ \text{subject to} & A(p_0) = 1, \end{array}$$

where

$$A(p) = \sum_{l \in \{\text{array elements}\}} w_l e^{-2\pi i p \cdot x_l}, \quad p \in S$$

w_l = complex-valued **design weight** for array element l

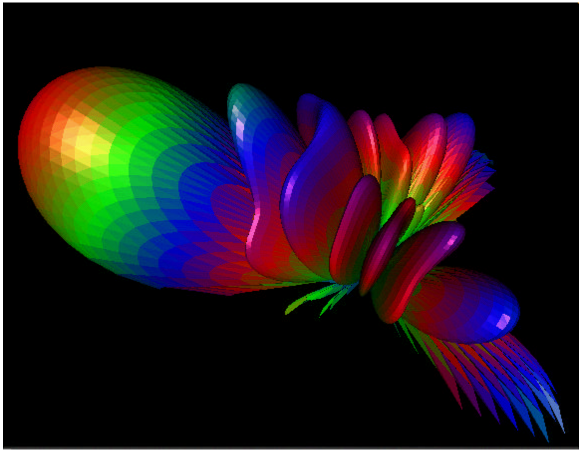
S = subset of unit hemisphere: sidelobe directions

x_l = spatial coord vector for array element l

p_0 = "look" direction

Specific Example: Hexagonal Lattice of 61 Elements

$$\begin{aligned} \rho &= -20 \text{ dB} = 0.01 \\ S &= 889 \text{ points outside } 20^\circ \text{ from look direction} \\ p_0 &= 40^\circ \text{ from zenith} \end{aligned}$$

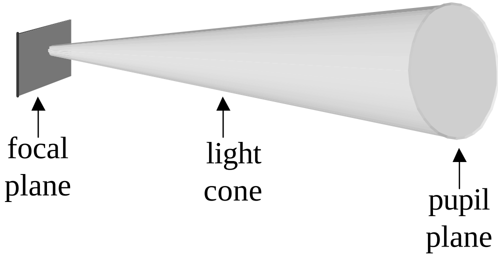


Shape Optimization (Telescope Design)

The problem is to design and build a space telescope that will be able to "see" planets around nearby stars (other than the Sun).

Consider a telescope. Light enters the front of the telescope. This is called the *pupil plane*.

The telescope focuses all the light passing through the pupil plane from a given direction at a certain point on the *focal plane*, say $(0, 0)$.



However, the wave nature of light makes it impossible to concentrate all of the light at a point. Instead, a small disk, called the *Airy disk*, with diffraction rings around it appears.

These diffraction rings are bright relative to any planet that might be orbiting a nearby star and so would completely hide the planet. The Sun, for example, would appear 10^{10} times brighter than the Earth to a distant observer.

By placing a mask over the pupil, one can design the shape and strength of the diffraction rings. The problem is to find an optimal shape so as to put a very deep *null* very close to the Airy disk.

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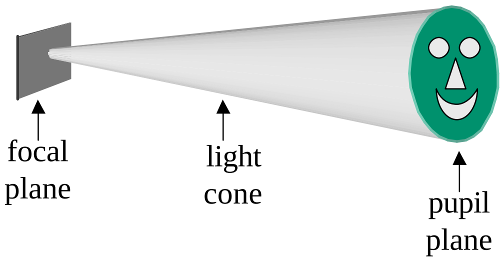

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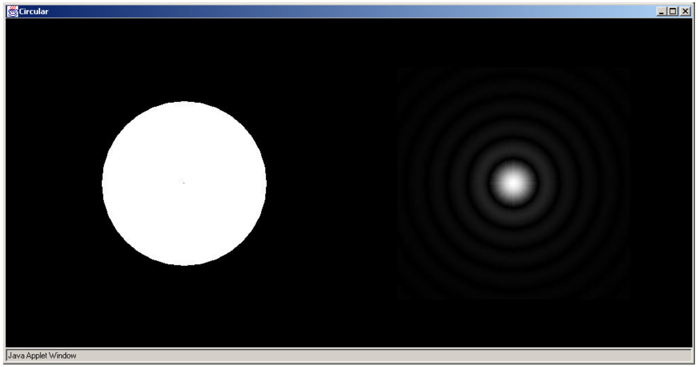
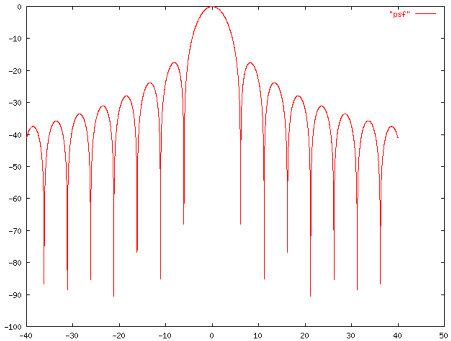
Airy Disk and Diffraction Rings

A conventional telescope has a circular openning as depicted by the left side of the figure. Visually, a star then looks like a small disk with rings around it, as depicted on the right.

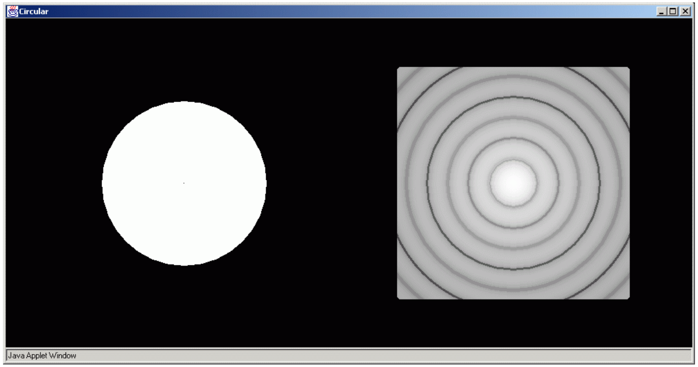


The rings grow progressively dimmer as this log-plot shows:



Airy Disk and Diffraction Rings—Log Scaling

Here's the same Airy disk from the previous slide plotted using a logarithmic brightness scale with $10^{-11} = -110$dB set to black:



The problem is to find an aperture mask, i.e. a pupil plane mask, that yields a -100 dB *null* somewhere near the first diffraction ring. *A hard problem!* Such a null would appear almost black in this log-scaled image.

Electric Field

Consider an aperture mask consisting of an openning given by

$$\left\{(x,y):\ -\frac{1}{2} \le x \le \frac{1}{2},\ -A(x) \le y \le A(x)\right\}.$$

We only consider masks that are symmetric with respect to both the x and y axes. Hence, the function $A()$ is a nonnegative even function.

In such a situation, the electric field $E(\xi,\zeta)$ is real and also symmetric about both the x and y axes. It is given by

$$\begin{aligned} E(\xi,\zeta) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-A(x)}^{A(x)} e^{i(x\xi+y\zeta)} dy dx \\ &= 4 \sum_j \int_0^{\frac{1}{2}} \cos(x\xi) \frac{\sin(A(x)\zeta)}{\zeta} dx \end{aligned}$$

The intensity of the light at (ξ,ζ) is given by the square of the electric field.

Maximizing Throughput

Because of the symmetry, we only need to optimize in the first quadrant:

$$\text{maximize } 4\int_0^{\frac{1}{2}} A(x)dx$$

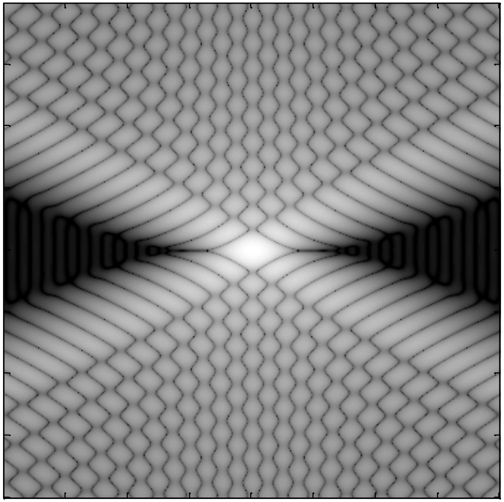
$$\begin{aligned}\text{subject to } & -10^{-5}E(0,0) \leq E(\xi,\zeta) \leq 10^{-5}E(0,0), && \text{for } (\xi,\zeta)\in\mathcal{O} \\ & 0 \leq A(x) \leq 1/2, && \text{for } 0 \leq x \leq 1/2\end{aligned}$$

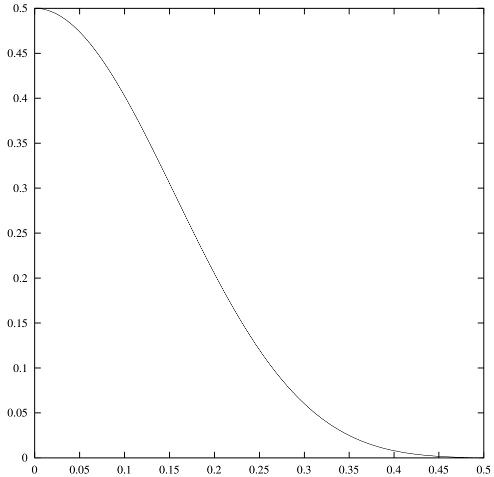
The objective function is the total open area of the mask. The first constraint guarantees 10^{-10} light intensity throughout a desired region of the focal plane, and the remaining constraint ensures that the mask is really a mask.

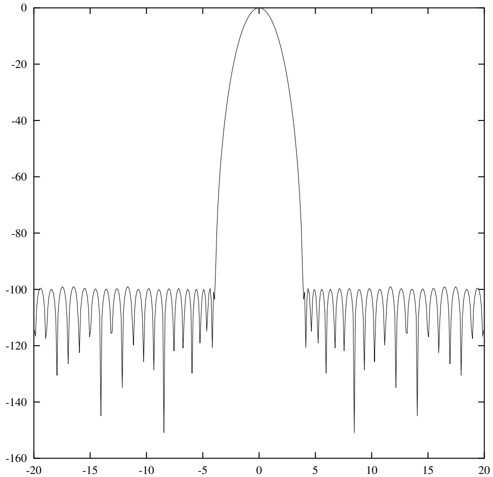
If the set $\mathcal{O}$ is a subset of the x-axis, then the problem is entirely linear (a linear programming problem).

One Pupil w/ On-Axis Constraints

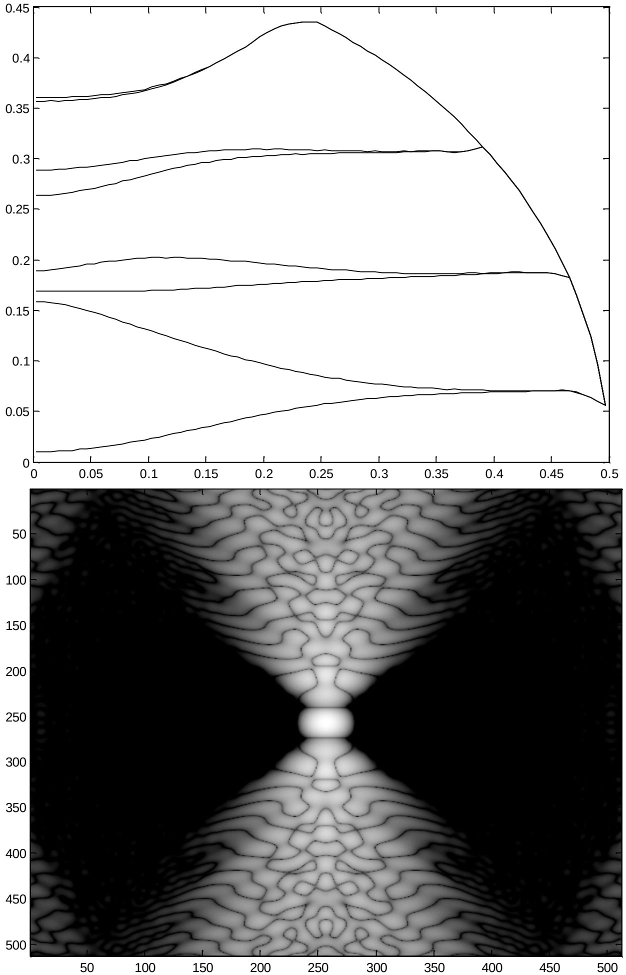
$\mathcal{O} = \{(\xi, 0) : \xi_0 \leq \xi \leq \xi_1\}$
$\xi_0 = 4$
$\xi_1 = 40$
Log scale:
white $= 0$dB,
black $= -110$dB
Thruput $= 37\%$







Best Mask



Home Page
Title Page
Contents
Page 20 of 32
Go Back
Full Screen
Close
Quit

Circularly Symmetric Masks

- My original question was "Why not work with circularly symmetric optics?" In this case, one could think of making a variable filter. That is, at point (x, y) have the filter transmit a fraction $A(x, y)$ of the light.
- Such a filter is called an *apodization*.
- The answer is that apodizations are hard to make *accurately*.
- For small working bands, the square-aperture masks are essentially bang-bang all-or-nothing masks.
- It suggests looking for similar circularly symmetric masks.
- They can be thought of as apodizations in which the apodizing function $A(r)$ is zero-one valued.
- On the next few slides we derive the formulas for circularly symmetric apodization and then restrict attention to the zero-one valued case.

Circularly Symmetric Apodization

Instead of a square mask, we consider now a circularly symmetric apodized aperture:

$$E(\xi,\zeta) = \int_0^{a/2}\int_{-\pi}^{\pi} A(r) e^{-2\pi i k(x\xi+y\zeta)/f} r\,d\theta\,dr$$

where, of course, $x = r\cos\theta$ and $y = r\sin\theta$.

WLOGWMAT, $\zeta = 0$ and hence we look at

$$\begin{aligned} E(\xi) &= \int_0^{a/2} rA(r)\left(\int_{-\pi}^{\pi} e^{-i\frac{2\pi k\xi}{f} r\cos\theta}\right) d\theta\,dr \\ &= \int_0^{a/2} 2\pi r A(r) J_0\left(\frac{2\pi k r\xi}{f}\right) dr \end{aligned}$$

Circularly Symmetric Masks

Let

$$A(r) = \begin{cases} 1 & r_{2j} \leq r \leq r_{2j+1}, \qquad j = 0, 1, \ldots, m-1 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$0 \leq r_0 \leq r_1 \leq \cdots \leq r_{2m-1} \leq a/2.$$

The integral on the previous slide can now be written as a sum of integrals and each of these integrals can be explicitly integrated to get:

$$E(\xi) = \sum_{j=0}^{m-1} \frac{f}{k\xi} \left(r_{2j+1} J_1 \left(\frac{2\pi k \xi r_{2j+1}}{f} \right) - r_{2j} J_1 \left(\frac{2\pi k \xi r_{2j}}{f} \right) \right).$$

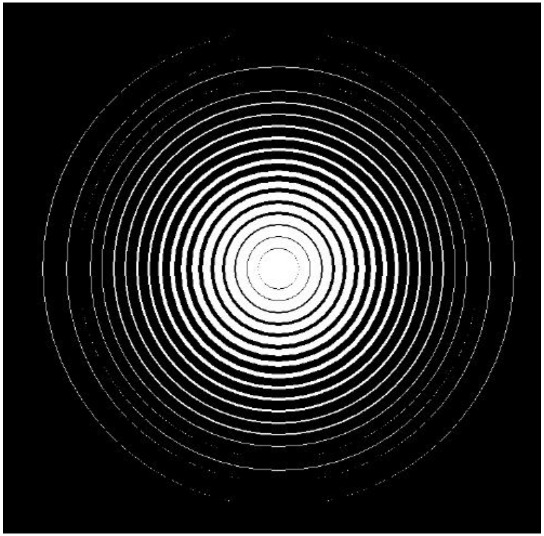
Circularly Symmetric Masks Optimization Problem

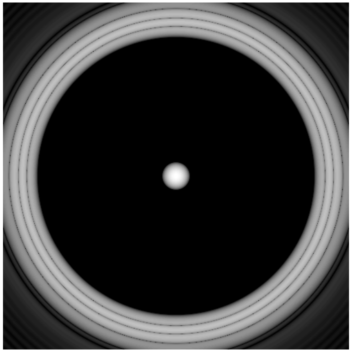
$$\text{maximize} \sum_{j=0}^{m-1} \pi(r_{2j+1}^2 - r_{2j}^2)$$

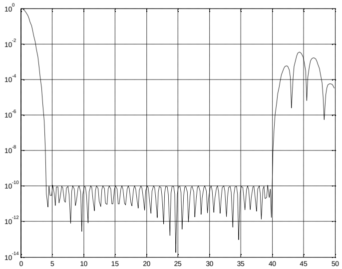
$$\text{subject to:} \quad -10^{-5}E(0) \le E(\xi) \le 10^{-5}E(0), \quad \text{for } \xi_0 \le \xi \le \xi_1$$

where $E(\xi)$ is the function of the r_j's given on the previous slide.

$\xi_0 = 4\lambda f/a$ and $\xi_1 = 48\lambda f/a$ and $m = 26$







Celestial Mechanics—Periodic Orbits

- Find periodic orbits for the planar gravitational n-body problem.
- Minimize action:

$$\int_0^{2\pi} (K(t) - P(t))dt,$$

- where $K(t)$ is kinetic energy,

$$K(t) = \frac{1}{2} \sum_i m_i \left(\dot{x}_i^2(t) + \dot{y}_i^2(t) \right),$$

- and $P(t)$ is potential energy,

$$P(t) = -\sum_{i<j} \frac{m_i m_j}{\sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2}}.$$

- Subject to periodicity constraints:

$$x_i(2\pi) = x_i(0), \qquad y_i(2\pi) = y_i(0).$$

Periodic Solutions

We assume solutions can be expressed in the form

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k^c \cos(kt) + a_k^s \sin(kt) \right)$$

$$y(t) = b_0 + \sum_{k=1}^{\infty} \left(b_k^c \cos(kt) + b_k^s \sin(kt) \right)$$

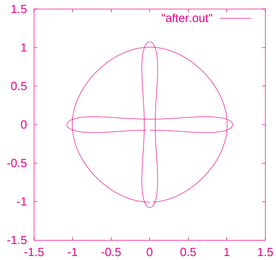
The variables a_0, a_k^c, a_k^s, b_0, b_k^c, and b_k^s are the *decision variables* in the optimization model.

Choreographies and the Ducati Orbit

Recently, Montgomery and Chencinier (2001) and then others discovered a host of new solutions to the equimass n-body problem. They call these solutions *choreographies* because all of the bodies follow the same path — they are simply spread out uniformly along this path. They found these orbits by *minimizing* the action functional.

I reproduced these choreographies using AMPL and LOQO and noticed that all except one are *unstable*.

This inspired me to look for stable solutions. I found a few, including the one that M. Todd called the *Ducati* solution.



Limitations of the Model

- The infinite sum gets truncated to a finite sum. This amounts to adding constraints. Hence, the solution might be suboptimal. That is, the trajectory obtained might not satisfy the equations of motion.
- Masses must be positive.
- Model can't solve 2-body problem w/ eccentricity (see next section).

Elliptic Solutions to the 2-Body Problem

An ellipse with semimajor axis a, semiminor axis b, and having its left focus at the origin of the coordinate system is given parametrically by:

$$x(t) = f + a\cos t, \qquad y(t) = b\sin t,$$

where $f = \sqrt{a^2 - b^2}$ is the distance from the focus to the center of the ellipse.

However, this is *not* the trajectory of a mass in the 2-body problem. Such a mass will travel faster around one focus than around the other. We need to introduce a time-change function $\theta(t)$:

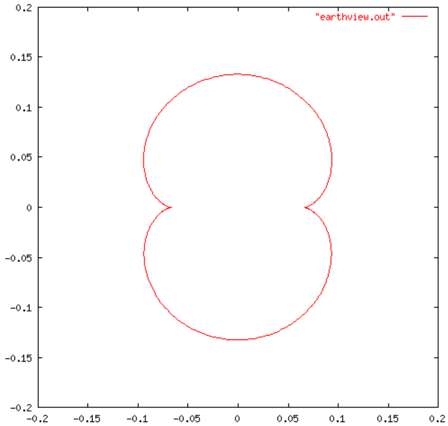
$$x(t) = f + a\cos\theta(t), \qquad y(t) = b\sin\theta(t).$$

This function θ must be increasing and must satisfy $\theta(0) = 0$ and $\theta(2\pi) = 2\pi$.

The optimization model can be used to find (a discretization of) $\theta(t)$ automatically by letting it be a vector of variables and adding appropriate monotonicity and boundary constraints.

A Hill-Type Solution to the Eccentric Sun-Earth System

Using an eccentricity $e = f/a = 0.0167$ and appropriate Sun and Earth masses, we can find a periodic Hill-Type satellite trajectory in which the satellite orbits the Earth once per year.



Contents