THE 2-BODY PROBLEM

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ABSTRACT. In this short note, we show that a pair of ellipses with a common focus is a solution to the 2-body problem

1. INTRODUCTION.

Solving the 2-body problem from scratch is doable but difficult. But, what if we simply want to verify that there are circular orbits where the two orbits share a common center? With these suppositions, maybe this problem isn't so hard. In other words, let's try to use the *guess-n-check* method. It should be easier. (We all believe that P $\neq$ NP, right!) Here we go...
First, choose the coordinate system so that the center of the orbits is at the center point, $(0, 0)$, of our coordinate system.
One of these two bodies, let's call it body 0, will be assumed to have a relatively large mass, let's call it M. The other body, let's call it body 1, will be assumed to have a smaller mass, let's call it m. The orbit of the two bodies can be given parametrically as

$$\begin{aligned} x_0 &= -r_0 \cos(2\pi t/T) & \qquad x_1 &= r_1 \cos(2\pi t/T) \\ y_0 &= -r_0 \sin(2\pi t/T). & \qquad y_1 &= r_1 \sin(2\pi t/T). \end{aligned}$$

Here, r_0, r_1 are the orbital radii (which are constants) and T is the orbital period (also a constant). The variables x_0, y_0, x_1 and y_1 are functions of time t.
The x and y coordinates of the center-of-mass of the system are:

$$\begin{aligned} x_{\text{ctr\_of\_mass}} &= Mx_0 + mx_1 & \qquad y_{\text{ctr\_of\_mass}} &= My_0 + my_1 \\ &= (-Mr_0 + mr_1)\cos(2\pi t/T), & &= (-Mr_0 + mr_1)\sin(2\pi t/T). \end{aligned}$$

The center of mass remains constant over time if and only if

$$-Mr_0 + mr_1 = 0$$

which implies this formula for r_0 in terms of r_1:

$$r_0 = \frac{m}{M} r_1.$$

If M is much larger than m, then r_0 will be very small which is the case when body 0 is our Sun and body 1 is Earth (or any other planet). Let's now focus on body 1. According to Newton's Law

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of Gravity, the strength of the gravitational force on body 1 is proportional to the product of the masses of the two bodies and is inversely proportional to the square of the distance between the two bodies. Let's start by computing the distance between the two bodies:

$$\begin{aligned}
\text{separation distance} &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
&= \sqrt{(r_1 + r_0)^2 \cos^2(2\pi t/T) + (y_1 - y_0)^2 \sin^2(2\pi t/T)} \\
&= \sqrt{(r_1 + r_0)^2} \\
&= r_1 + r_0.
\end{aligned}$$

If we let $|F|$ denote the magnitude of the force and G denote the constant of proportionality, we have

$$|F| = \frac{GmM}{(r_1 + r_0)^2}$$

The actual force, F, is a vector that points from body 1 to body 0. So, we need to multiply $|F|$ by a unit vector pointing in this direction. Since, from body 1's perspective, body 2 lies on the other side of the origin $(0, 0)$, we can take the negative of body 1's "position" vector and normalize it to a unit vector by dividing by its length, r_1:

$$\begin{aligned}
F = (F_x, F_y) &= -|F| \left(\cos(2\pi t/T), \sin(2\pi t/T)\right) & (1) \\
&= -\frac{GmM}{(r_1 + r_0)^2} \left(\cos(2\pi t/T), \sin(2\pi t/T)\right) & (2)
\end{aligned}$$

Now, according to Newton's 2nd Law of Motion, the force is equal to the mass times the acceleration. So, for the x-component of the force, we get that

$$F_x = m \frac{d^2}{dt^2} x_1(t). \tag{3}$$

So, we need to differentiate twice. Using the chain rule, the first derivative is:

$$\frac{d}{dt} x_1(t) = -r_1 \sin(2\pi t/T) 2\pi/T.$$

And, hence, the second derivative is:

$$\frac{d^2}{dt^2} x_1(t) = -r_1 \cos(2\pi t/T) 4\pi^2/T^2.$$

Plugging this formula and the formula for F_x from equation (2) into equation 3, we get

$$-\frac{GmM}{(r_1 + r_0)^2} \cos(2\pi t/T) = -m r_1 \cos(2\pi t/T) 4\pi^2/T^2.$$

It is easy to check that the y-component of force leads to the same equation with the cosine replaced by a sine. But, the next step is to notice that the cosine appears on both sides of the equation and so we can divide it out. We can also divide both sides by $-m$. After these divisions, we get:

$$\frac{GM}{(r_1 + r_0)^2} = r_1 4\pi^2/T^2.$$

We're almost done. What need to recall that $r_0 = (m/M)r_1$. Substituting this into our formula, we get

$$\frac{GM}{(r_1 + (m/M)r_1)^2} = r_1 4\pi^2/T^2.$$

After some algebraic rearrangements, we get

$$\frac{GM^3}{r_1^3} = 4\pi^2(M+m)^2/T^2.$$

Or, rearranging it differently, we see that

$$\frac{r_1^3}{T^2} = \frac{GM^3}{4\pi^2(M+m)^2}.$$

Finally, if we assume that M is much larger than m, then we can make a simple approximation that $M \approx M + m$ and arrive at our final formula:

$$r_1^3 = \frac{GM}{4\pi^2} T^2$$

If this relation between r_1 and T (and r_2 and T) is satisfied, then the circular orbits due indeed satisfy Newton's laws. This last formula is called *Kepler's Third Law of Planetary Motion*.

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