THE 2-BODY PROBLEM

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ABSTRACT. In this short note, we show that a pair of ellipses with a common focus is a solution to the 2-body problem.

1. INTRODUCTION.

Solving the 2-body problem from scratch is doable but difficult. But, what if we simply want to verify that there are elliptical orbits where the two ellipses share a common focus, which is also the center of mass of the system? With these suppositions, maybe this problem isn’t so hard to solve. In other words, let’s try to use the so-called guess-n-check method. It should be easier. (We all believe that \( P \neq NP \), right!) Here we go...

First, choose the coordinate system so that the foci lie on the horizontal \( x \)-axis and so that the shared focus is at the origin (see Figure 1).

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FIGURE 1. A pair of ellipses sharing a common focus.
One body’s second focus is on the positive $x$-axis and the other body’s second focus is on the negative $x$-axis. Call the right body “body one” and the left body will be “body two”. The orbit of body one can be given parametrically as

$$x_1 = c + a \cos \theta,$$
$$y_1 = b \sin \theta.$$  

Here, $a$, $b$, and $c$ are constants whereas $x_1$, $y_1$, and $\theta$ are functions of time $t$. The constant $c$ is the $x$-coordinate of the center of the ellipse and the constants $a$ and $b$ are the semi-major and semi-minor axes, respectively. Clearly all three constants are positive numbers and $a > b$. Furthermore, an important property of ellipses is that the distance from the center of the ellipse to a focus is $\sqrt{a^2 - b^2}$. Since $(c, 0)$ is the center of the ellipse and $(0, 0)$ is a focus, it follows that

$$c = \sqrt{a^2 - b^2}.$$  

The assumption that the center of mass of the system coincides with the focus at the origin implies that

$$m_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = -m_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$  

Here, $m_1$ and $m_2$ are the masses of the two bodies—they need not be the same. The distance $r$ between the two bodies plays and important role in Newton’s law of gravity, so we start by computing it:

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$  

$$= \sqrt{\left(\frac{m_1}{m_2} + 1\right)^2 x_1^2 + \left(\frac{m_1}{m_2} + 1\right)^2 y_1^2}$$  

$$= \left(\frac{m_1}{m_2} + 1\right) \sqrt{x_1^2 + y_1^2}$$  

$$= \left(\frac{m_1}{m_2} + 1\right) \sqrt{(c + a \cos \theta)^2 + (b \sin \theta)^2}$$  

$$= \left(\frac{m_1}{m_2} + 1\right) \sqrt{c^2 + 2ac \cos \theta + a^2 \cos^2 \theta + b^2 \sin^2 \theta}$$  

$$= \left(\frac{m_1}{m_2} + 1\right) \sqrt{c^2 + 2ac \cos \theta + a^2 \cos^2 \theta + \left(a^2 - c^2\right) \sin^2 \theta}$$  

$$= \left(\frac{m_1}{m_2} + 1\right) \sqrt{c^2 \cos^2 \theta + 2ac \cos \theta + a^2 - c^2 \sin^2 \theta}$$  

$$= \left(\frac{m_1}{m_2} + 1\right) \left(a + c \cos \theta\right).$$
Newton’s laws involve accelerations and so we differentiate once

\[ \begin{align*}
\dot{x}_1 &= -a \sin \theta \ \dot{\theta} \\
\dot{y}_1 &= b \cos \theta \ \dot{\theta}
\end{align*} \]

and then a second time

\[ \begin{align*}
\ddot{x}_1 &= -a \sin \theta \ddot{\theta} - a \cos \theta \dot{\theta}^2 \\
\ddot{y}_1 &= b \cos \theta \ddot{\theta} - b \sin \theta \dot{\theta}^2.
\end{align*} \]

Using Newton’s law of gravity together with Newton’s second law of motion we see that

\[ \begin{align*}
m_1 \ddot{x}_1 &= G m_1 m_2 \frac{x_2 - x_1}{r^3} = G m_1 m_2 \frac{-(m_1/m_2 + 1)x_1}{(m_1/m_2 + 1)(a + c \cos \theta)^3} \\
m_1 \ddot{y}_1 &= G m_1 m_2 \frac{y_2 - y_1}{r^3} = G m_1 m_2 \frac{-(m_1/m_2 + 1)y_1}{(m_1/m_2 + 1)(a + c \cos \theta)^3}.
\end{align*} \]

Simplifying, we get

\[ \begin{align*}
\ddot{x}_1 &= -G \mu_2^2 m_2 \frac{x_1}{(a + c \cos \theta)^3} \\
\ddot{y}_1 &= -G \mu_2^2 m_2 \frac{y_1}{(a + c \cos \theta)^3},
\end{align*} \]

where \( \mu_2 = m_2/(m_1 + m_2) \). Equating the two formulas for the second derivatives derived above, we get

\[ \begin{align*}
-a \sin \theta \ddot{\theta} - a \cos \theta \dot{\theta}^2 &= -G \mu_2^2 m_2 \frac{(c + a \cos \theta)}{(a + c \cos \theta)^3} \\
b \cos \theta \ddot{\theta} - b \sin \theta \dot{\theta}^2 &= -G \mu_2^2 m_2 \frac{b \sin \theta}{(a + c \cos \theta)^3}.
\end{align*} \]

Negating both sides and dividing the two equations by \( a \) and by \( b \) and then writing in matrix form, we see that

\[
\begin{bmatrix}
\sin \theta & \cos \theta \\
-cos \theta & \sin \theta
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta} \\
\dot{\theta}^2
\end{bmatrix}
= \begin{bmatrix}
G \mu_2^2 m_2 \frac{c/a + \cos \theta}{(a + c \cos \theta)^3} \\
G \mu_2^2 m_2 \frac{\sin \theta}{(a + c \cos \theta)^3}
\end{bmatrix}.
\]
Multiplying by the matrix inverse, we get
\[
\begin{bmatrix}
\ddot{\theta} \\
\dot{\theta}^2
\end{bmatrix} = G\mu_2^2 m_2 \begin{bmatrix}
\sin \theta & -\cos \theta \\
\cos \theta & \sin \theta
\end{bmatrix} \begin{bmatrix}
c/a + \cos \theta \\
\sin \theta
\end{bmatrix} \frac{1}{(a + c \cos \theta)^3}
\]
\[
= G\mu_2^2 m_2 \begin{bmatrix}
\sin \theta (c/a + \cos \theta) - \cos \theta \sin \theta \\
\cos \theta (c/a + \cos \theta) + \sin \theta \sin \theta
\end{bmatrix} \frac{1}{(a + c \cos \theta)^3}
\]
\[
= G\mu_2^2 m_2 \begin{bmatrix}
(c/a) \sin \theta \\
(c/a) \cos \theta + 1
\end{bmatrix} \frac{1}{(a + c \cos \theta)^3}
\]
\[
= \begin{bmatrix}
G\mu_2^2 m_2 \frac{c \sin \theta}{a(a + c \cos \theta)^3} \\
G\mu_2^2 m_2 \frac{1}{a(a + c \cos \theta)^2}
\end{bmatrix}
\]

In other words, for a solution to exist, the function \( \theta \) must simultaneously be a solution to both of the following, possibly inconsistent, equations:
\[
\ddot{\theta} = G\mu_2^2 m_2 \frac{c \sin \theta}{a(a + c \cos \theta)^3}
\]
\[
\dot{\theta} = \sqrt{Gm_2\mu_2} \frac{1}{\sqrt{a(a + c \cos \theta)}}
\]

To verify consistency, we check that differentiating the equation for \( \dot{\theta} \) brings us to the equation derived above for \( \ddot{\theta} \):
\[
\frac{d}{dt} \dot{\theta} = \sqrt{Gm_2\mu_2} \frac{1}{\sqrt{a}} \frac{c \sin \theta \dot{\theta}}{(a + c \cos \theta)^2}
\]
\[
= \sqrt{Gm_2\mu_2} \frac{1}{\sqrt{a}} \frac{c \sin \theta}{(a + c \cos \theta)^2} \sqrt{Gm_2\mu_2} \frac{1}{\sqrt{a(a + c \cos \theta)}}
\]
\[
= Gm_2\mu_2^2 \frac{c \sin \theta}{a(a + c \cos \theta)^3}
\]

Hence, we see that the formulas for the first and second derivative of \( \theta \) are indeed consistent with each other.

Picking \( \theta(0) \) can be given any value as it merely determines where the bodies are in their orbit at time \( t = 0 \). The semi-major and semi-minor axes \( a \) and \( b \) determine \( c \) (as already mentioned) and \( \dot{\theta}(0) \):
\[
\dot{\theta}(0) = G\mu_2^2 m_2 \frac{c \sin \theta(0)}{a(a + c \cos \theta(0))^3}
\]
Suppose, for simplicity, that $\theta(0) = 0$. The differential equation for $\dot{\theta}(t)$ can be rewritten as

$$(a + c \cos \theta) \, d\theta = \sqrt{Gm_2\mu_2} \, \frac{dt}{\sqrt{a}}.$$

Since $\varepsilon = c/a$ is the well-known eccentricity of the ellipse, it is common to divide both sides of this equation by $a$ thereby replacing $c$ with the eccentricity $\varepsilon$:

$$(1 + \varepsilon \cos \theta) \, d\theta = \sqrt{Gm_2\mu_2} \, \frac{dt}{a^{3/2}}.$$

Integrating from 0 to $t$, we get

$$\int_0^{\theta(t)} (1 + \varepsilon \cos \theta) \, d\theta = \int_0^t \sqrt{Gm_2\mu_2} \, \frac{dt}{a^{3/2}}.$$

The integrals can be computed explicitly. The result is

$$\theta(t) + \varepsilon \sin \theta(t) = \sqrt{Gm_2\mu_2} \frac{t}{a^{3/2}}.$$

Unfortunately, this is a transcendental equation for $\theta(t)$ and so it does not have a simple “closed form” solution. However, in the case where the eccentricity is zero, we have an exact solution:

$$\theta(t) = \sqrt{Gm_2\mu_2} \frac{t}{a^{3/2}}.$$

And, when the eccentricity is not zero, the following recursion quickly converges on the correct answer:

$$\begin{align*}
\theta^{(0)}(t) &= \sqrt{Gm_2\mu_2} \frac{t}{a^{3/2}} \\
\theta^{(1)}(t) &= \sqrt{Gm_2\mu_2} \frac{t}{a^{3/2}} - \varepsilon \sin \theta^{(0)}(t) \\
&\vdots \\
\theta^{(k+1)}(t) &= \sqrt{Gm_2\mu_2} \frac{t}{a^{3/2}} - \varepsilon \sin \theta^{(k)}(t) \\
&\vdots
\end{align*}$$

By deriving these equations we have shown that there are elliptical orbits that are solutions to the 2-body problem. This is the essence of Kepler’s First Law of Planetary Motion. Kepler’s Second Law of Planetary Motion tells us that the “area” of an orbit grows linearly with time. Let’s check that by defining the area generated by the orbit around the center of mass, which is at $(0, 0)$ in our coordinate system. If the think of an infinitesimally short interval of time, $dt$, the area swept out by body 1 in that small interval of time is equal to the distance that that body is from the center of mass times the angular change of position over that small time interval:

$$\dot{A} = r \dot{\theta}.$$
Here, $A$ is the area as a function of time and $r$ is the distance from the center of mass to the body. Let’s compute $r$:

\[
r = \sqrt{x^2 + y^2} = \sqrt{(c + a \cos(\theta))^2 + b^2 \sin^2(\theta)} = \sqrt{c^2 + 2ac \cos(\theta) + a^2 \cos^2(\theta) + b^2 \sin^2(\theta)} = \sqrt{a^2 - b^2 + 2ac \cos(\theta) + a^2 \cos^2(\theta) + b^2 \sin^2(\theta)} = \sqrt{a^2 + 2ac \cos(\theta) + a^2 \cos^2(\theta) - b^2 (1 - \sin^2(\theta))} = \sqrt{a^2 + 2ac \cos(\theta) + a^2 \cos^2(\theta) - b^2 \cos^2(\theta)} = a + c \cos(\theta)
\]

Now, use the formula we derived earlier for $\dot{\theta}$ to compute $\dot{A}$:

\[
\dot{A} = r \dot{\theta} = (a + c \cos(\theta)) \sqrt{\frac{G m_2 \mu_2}{a(a + c \cos(\theta))}} \frac{1}{\sqrt{a(a + c \cos(\theta))}} = \sqrt{\frac{G m_2}{a}} \mu_2
\]

The formula does not depend on time and therefore Kepler’s second law is now proved.