KEPLER'S LAWS FOR THE 2-BODY PROBLEM

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ABSTRACT. Kepler's three laws of planetary motion describe the dynamics of the 2-body problem where one body is the Sun and the other body is a planet. The Sun is way more massive than all of our planets, even Jupiter, and therefore the Kepler's original derivations are about the 2-body problem where the mass ratio is infinite. In this short note, we'll show that the same three laws apply no matter what the mass ratios are.

1. KEPLER'S FIRST LAW – ELLIPTICAL ORBITS.

Solving the 2-body problem from scratch is doable but difficult. But, what if we simply want to verify that there are elliptical orbits where the two ellipses share a common focus, which is also the center of mass of the system? With these suppositions, maybe this problem isn't so hard to

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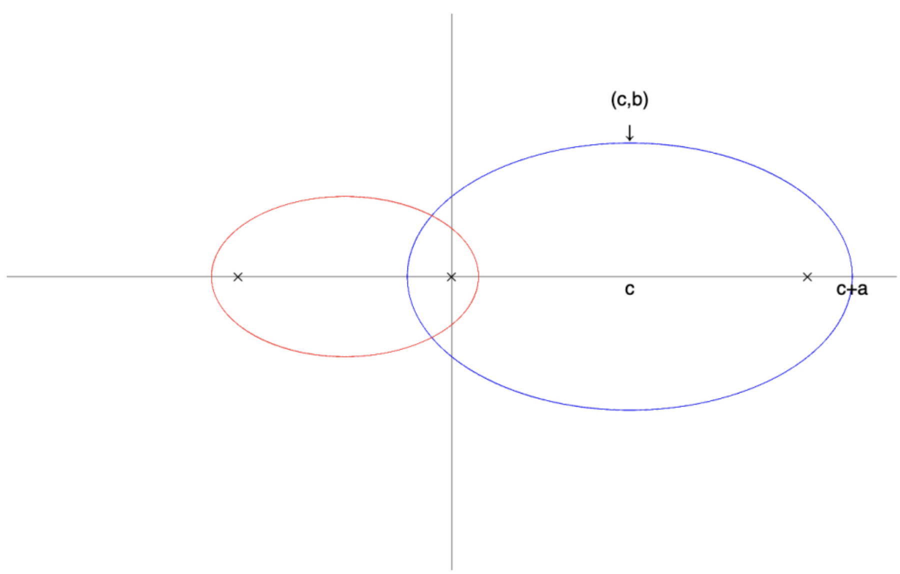

FIGURE 1. A pair of ellipses sharing a common focus.

solve. In other words, let's try to use the so-called *guess-n-check* method. It should be easier. (We all believe that P $\neq$ NP, right!) Here we go...

First, choose the coordinate system so that the foci lie on the horizontal x-axis and so that the shared focus is at the origin (see Figure 1).
One body's second focus is on the positive x-axis and the other body's second focus is on the negative x-axis. Call the right body "body one" and the left body will be "body two". The orbit of body one can be given parametrically as

$$\begin{aligned} x_1 &= c + a\cos\theta \\ y_1 &= b\sin\theta. \end{aligned}$$

Here, a, b, and c are constants whereas x_1, y_1, and θ are functions of time t. The constant c is the x-coordinate of the center of the ellipse and the constants a and b are the semi-major and semi-minor axes, respectively. Clearly all three constants are positive numbers and $a > b$. Furthermore, an important property of ellipses is that the distance from the center of the ellipse to a focus is $\sqrt{a^2 - b^2}$. Since $(c, 0)$ is the center of the ellipse and $(0, 0)$ is a focus, it follows that

$$c = \sqrt{a^2 - b^2}.$$

The assumption that the center of mass of the system coincides with the focus at the origin implies that

$$m_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = -m_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Here, m_1 and m_2 are the masses of the two bodies—they need not be the same.

The distance r between the two bodies plays and important role in Newton's law of gravity, so we start by computing it:

$$\begin{aligned}
r &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{\left(-\left(\frac{m_1}{m_2} + 1\right) x_1\right)^2 + \left(-\left(\frac{m_1}{m_2} + 1\right) y_1\right)^2} \\
&= \left(\frac{m_1}{m_2} + 1\right) \sqrt{x_1^2 + y_1^2} \\
&= \left(\frac{m_1}{m_2} + 1\right) \sqrt{(c + a\cos\theta)^2 + (b\sin\theta)^2} \\
&= \left(\frac{m_1}{m_2} + 1\right) \sqrt{c^2 + 2ac\ \cos\theta + a^2\cos^2\theta + b^2\sin^2\theta} \\
&= \left(\frac{m_1}{m_2} + 1\right) \sqrt{c^2 + 2ac\ \cos\theta + a^2\cos^2\theta + (a^2 - c^2)\sin^2\theta} \\
&= \left(\frac{m_1}{m_2} + 1\right) \sqrt{c^2\cos^2\theta + 2ac\ \cos\theta + a^2} \\
&= \left(\frac{m_1}{m_2} + 1\right) (a + c\ \cos\theta).
\end{aligned}$$

Newton's laws involve accelerations and so we differentiate once

$$\begin{aligned}
\dot{x}_1 &= -a\sin\theta\ \dot{\theta} \\
\dot{y}_1 &= b\cos\theta\ \dot{\theta}
\end{aligned}$$

and then a second time

$$\begin{aligned}
\ddot{x}_1 &= -a\sin\theta\ \ddot{\theta} - a\cos\theta\ \dot{\theta}^2 \\
\ddot{y}_1 &= b\cos\theta\ \ddot{\theta} - b\sin\theta\ \dot{\theta}^2.
\end{aligned}$$

Using Newton's law of gravity together with Newton's second law of motion we see that

$$\begin{aligned}
m_1\ddot{x}_1 &= Gm_1m_2\frac{x_2 - x_1}{r^3} &= Gm_1m_2\frac{-(m_1/m_2 + 1)x_1}{(m_1/m_2 + 1)^3(a + c\ \cos\theta)^3} \\
m_1\ddot{y}_1 &= Gm_1m_2\frac{y_2 - y_1}{r^3} &= Gm_1m_2\frac{-(m_1/m_2 + 1)y_1}{(m_1/m_2 + 1)^3(a + c\ \cos\theta)^3}.
\end{aligned}$$

Simplifying, we get

$$\begin{aligned}
\ddot{x}_1 &= -G\mu_2^2 m_2\frac{x_1}{(a + c\ \cos\theta)^3} \\
\ddot{y}_1 &= -G\mu_2^2 m_2\frac{y_1}{(a + c\ \cos\theta)^3},
\end{aligned}$$

where $\mu_2 = m_2/(m_1+m_2)$. Equating the two formulas for the second derivatives derived above, we get

$$\begin{aligned} -a\sin\theta\,\ddot\theta - a\cos\theta\,\dot\theta^2 &= -G\mu_2^2 m_2 \frac{(c+a\cos\theta)}{(a+c\ \cos\theta)^3} \\ b\cos\theta\,\ddot\theta - b\sin\theta\,\dot\theta^2 &= -G\mu_2^2 m_2 \frac{b\sin\theta}{(a+c\ \cos\theta)^3}. \end{aligned}$$

Negating both sides and dividing the two equations by a and by b and then writing in matrix form, we see that

$$\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \ddot\theta \\ \dot\theta^2 \end{bmatrix} = \begin{bmatrix} G\mu_2^2 m_2 \dfrac{c/a+\cos\theta}{(a+c\ \cos\theta)^3} \\ G\mu_2^2 m_2 \dfrac{\sin\theta}{(a+c\ \cos\theta)^3} \end{bmatrix}$$

Multiplying by the matrix inverse, we get

$$\begin{aligned} \begin{bmatrix} \ddot\theta \\ \dot\theta^2 \end{bmatrix} &= G\mu_2^2 m_2 \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \dfrac{c/a+\cos\theta}{(a+c\ \cos\theta)^3} \\ \dfrac{\sin\theta}{(a+c\ \cos\theta)^3} \end{bmatrix} \\ &= \frac{G\mu_2^2 m_2}{(a+c\ \cos\theta)^3} \begin{bmatrix} \sin\theta\ (c/a+\cos\theta) - \cos\theta\sin\theta \\ \cos\theta\ (c/a+\cos\theta) + \sin\theta\sin\theta \end{bmatrix} \\ &= \frac{G\mu_2^2 m_2}{(a+c\ \cos\theta)^3} \begin{bmatrix} (c/a)\sin\theta \\ (c/a)\cos\theta + 1 \end{bmatrix} \\ &= \begin{bmatrix} G\mu_2^2 m_2 \dfrac{c\sin\theta}{a(a+c\ \cos\theta)^3} \\ G\mu_2^2 m_2 \dfrac{1}{a(a+c\ \cos\theta)^2} \end{bmatrix} \end{aligned}$$

In other words, for a solution to exist, the function θ must simultaneously be a solution to both of the following, possibly inconsistent, equations:

$$\begin{aligned} \ddot\theta &= G\mu_2^2 m_2 \frac{c\sin\theta}{a(a+c\ \cos\theta)^3} \\ \dot\theta &= \sqrt{Gm_2}\mu_2 \frac{1}{\sqrt{a}(a+c\ \cos\theta)} \end{aligned}$$

To verify consistency, we check that differentiating the equation for $\dot{\theta}$ brings us to the equation derived above for $\ddot{\theta}$:

$$\begin{aligned} \frac{d}{dt}\dot{\theta} &= \sqrt{Gm_2}\mu_2 \frac{1}{\sqrt{a}} \; \frac{c\ \sin\theta\ \dot{\theta}}{(a+c\ \cos\theta)^2} \\ &= \sqrt{Gm_2}\mu_2 \frac{1}{\sqrt{a}} \; \frac{c\ \sin\theta}{(a+c\ \cos\theta)^2} \; \sqrt{Gm_2}\mu_2 \frac{1}{\sqrt{a}(a+c\ \cos\theta)} \\ &= Gm_2\mu_2^2 \frac{c\sin\theta}{a(a+c\ \cos\theta)^3}. \end{aligned}$$

Hence, we see that the formulas for the first and second derivative of θ are indeed consistent with each other.
The angular orientation at time zero, $\theta(0)$, can be given any value as it merely determines where the bodies are in their orbit at time $t = 0$. The semi-major and semi-minor axes a and b determine c (as already mentioned) and $\dot{\theta}(0)$:

$$\dot{\theta}(0) = \sqrt{Gm_2}\mu_2 \frac{1}{\sqrt{a}(a+c\ \cos\theta(0))}.$$

Suppose, for simplicity, that $\theta(0) = 0$. The differential equation for $\dot{\theta}(t)$ can be rewritten as

$$(a + c\cos\theta)\, d\theta = \sqrt{Gm_2}\mu_2 \frac{dt}{\sqrt{a}}.$$

Since $\varepsilon = c/a$ is the well-known *eccentricity* of the ellipse, it is common to divide both sides of this equation by a thereby replacing c with the eccentricity ε:

$$(1 + \varepsilon\cos\theta)\, d\theta = \sqrt{Gm_2}\mu_2 \frac{dt}{a^{3/2}}.$$

Integrating from 0 to t, we get

$$\int_0^{\theta(t)} (1 + \varepsilon\cos\theta)\, d\theta = \int_0^t \sqrt{Gm_2}\mu_2 \frac{dt}{a^{3/2}}.$$

The integrals can be computed explicitly. The result is

$$\theta(t) + \varepsilon\sin\theta(t) = \sqrt{Gm_2}\mu_2 \frac{t}{a^{3/2}}.$$

Unfortunately, this is a transcendental equation for $\theta(t)$ and so it does not have a simple "closed form" solution. However, in the case where the eccentricity is zero, we have an exact solution:

$$\theta(t) = \sqrt{Gm_2}\mu_2 \frac{t}{a^{3/2}}.$$

And, when the eccentricity is not zero, the following recursion quickly converges on the correct answer:

$$\begin{aligned}
\theta^{(0)}(t) &= \sqrt{Gm_2}\mu_2 \frac{t}{a^{3/2}} \\
\theta^{(1)}(t) &= \sqrt{Gm_2}\mu_2 \frac{t}{a^{3/2}} - \varepsilon \sin\theta^{(0)}(t) \\
&\vdots \\
\theta^{(k+1)}(t) &= \sqrt{Gm_2}\mu_2 \frac{t}{a^{3/2}} - \varepsilon \sin\theta^{(k)}(t) \\
&\vdots
\end{aligned}$$

By deriving these equations we have shown that there are elliptical orbits that are solutions to the 2-body problem. This is the essence of Kepler's *First Law of Planetary Motion*.

2. Kepler's Second Law – Constant Area per Time.

Kepler's *Second Law of Planetary Motion* tells us that the "area" swept out by an orbit grows linearly with time. Let's check that by defining the area generated by the orbit around the center of mass, which is at $(0, 0)$ in our coordinate system. Let's let $A(t)$ denote the area swept out by the orbit of body 1 from time 0 to t. If the think of an infinitesimally short interval of time, dt, the area swept out by body 1 in that small interval of time is equal to the distance that that body is from the center of mass times the angular change of position over that small time interval. The formula is easy to express in polar coordinates. But, unfortunately, θ as we have defined it is not the polar coordinate angle. It's just the angular parameter in the formula for our ellipse. So, let's let our polar coordinates be r and ϕ. Using these coordinates, the area swept out in a small interval of time, dt, is given by

$$\dot{A} = \frac{1}{2} r^2 \dot{\phi}.$$

We now need to derive appropriate formulas for r and $\dot{\phi}$, where r is the distance from the center of mass to body number 1. Let's compute r:

$$\begin{aligned} r &= \sqrt{x_1^2 + y_1^2} \\ &= \sqrt{(c + a\cos(\theta))^2 + b^2\sin^2(\theta)} \\ &= \sqrt{c^2 + 2ac\cos(\theta) + a^2\cos^2(\theta) + b^2\sin^2(\theta)} \\ &= \sqrt{a^2 - b^2 + 2ac\cos(\theta) + a^2\cos^2(\theta) + b^2\sin^2(\theta)} \\ &= \sqrt{a^2 + 2ac\cos(\theta) + a^2\cos^2(\theta) - b^2(1 - \sin^2(\theta))} \\ &= \sqrt{a^2 + 2ac\cos(\theta) + a^2\cos^2(\theta) - b^2\cos^2(\theta)} \\ &= \sqrt{a^2 + 2ac\cos(\theta) + c^2\cos^2(\theta)} \\ &= a + c\cos(\theta) \end{aligned}$$

To compute $\dot{\phi}$, let's start by writing the formulas for the sine and cosine of ϕ. The sine function is defined as the "opposite over hypotenuse":

$$\sin\phi = \frac{y_1}{r} = \frac{b\sin\theta}{a + c\cos\theta}$$

and the cosine function is defined as the "adjacent over hypotenuse":

$$\cos\phi = \frac{x_1}{r} = \frac{c + a\cos\theta}{a + c\cos\theta}.$$

We need a formula for $\dot{\phi}$ and so let's differentiate the sine equation with respect to time:

$$\begin{aligned} \cos\phi\,\dot{\phi} &= \frac{(a + c\cos\theta)b\cos\theta\,\dot{\theta} \;+\; b\sin\theta\,c\sin\theta\,\dot{\theta}}{(a + c\cos\theta)^2} \\ &= b\frac{a\cos\theta + c}{(a + c\cos\theta)^2}\,\dot{\theta}. \end{aligned}$$

Plugging in our formula for $\cos(\phi)$, we get

$$\frac{c + a\cos\theta}{a + c\cos\theta}\,\dot{\phi} = b\frac{a\cos\theta + c}{(a + c\cos\theta)^2}\,\dot{\theta}.$$

And now let's solve this for $\dot{\phi}$:

$$\begin{aligned} \dot{\phi} &= \frac{b}{a + c\cos\theta}\,\dot{\theta} \\ &= \frac{b}{r}\,\dot{\theta}. \end{aligned}$$

Now, use the formula we derived earlier for $\dot{\theta}$ to compute $\dot{A}$:

$$\begin{aligned}\dot{A} &= \frac{1}{2}r^2\dot{\phi} \\ &= \frac{b}{2}r\dot{\theta} \\ &= \frac{b}{2}(a + c\cos(\theta))\sqrt{Gm_2}\mu_2\frac{1}{\sqrt{a}(a + c\ \cos\theta)} \\ &= \frac{b}{2}\sqrt{\frac{Gm_2}{a}}\ \mu_2\end{aligned}$$

The formula does not depend on time and therefore Kepler's second law is now proved.

3. Kepler's Third Law – Orbital Period.

Kepler's third law tells us that the orbital period of a planet is proportional to the square root of the cube of the semi-major axis. Let's see if that's true in the current setting.

From Kepler's second law we see that

$$A(t) = \frac{b}{2}\sqrt{\frac{Gm_2}{a}}\ \mu_2\ t.$$

And, from Kepler's first law, we know that the orbit is an ellipse with semi-major axis a and semi-minor axis b. The area of such an ellipse is πab. Let T denote the orbital period so that we can write:

$$A(T) = \frac{b}{2}\sqrt{\frac{Gm_2}{a}}\ \mu_2\ T = \pi ab.$$

Solving for T, we get

$$\begin{aligned}T &= \frac{2\pi a}{\sqrt{\frac{Gm_2}{a}}\ \mu_2} \\ &= \frac{2\pi}{\sqrt{Gm_2}\ \mu_2}a^{3/2}\end{aligned}$$

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