Linear Stability of Lagrange Points: Complex Variable Notation

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ABSTRACT

Subject headings: Celestial mechanics, Lagrange points, linear stability

1. Introduction

Consider the planar 3-body problem. Number the bodies $0, \ldots, 2$ and let m_j denote the mass of body j. Without loss of generality we may assume that $m_0 \geq m_1 \geq m_2$ in which case we often refer to body 0 as the Sun, body 1 as a planet, and body 2 as a small third mass (a satellite, an asteroid, etc.).

Let $z_j = x_j + iy_j$ denote the position of body j represented as a point in the complex plane $\mathbb{C}$. According to Newton's law of gravitation, the gravitational force acting on body j is given by

$$F_j = Gm_j \sum_{k \neq j} m_k \frac{z_k - z_j}{|z_k - z_j|^3}.$$

Of course, the force is only relevant in that it determines the acceleration

$$\ddot{z}_j = \frac{F_j}{m_j} = G \sum_{k \neq j} m_k \frac{z_k - z_j}{|z_k - z_j|^3}. \tag{1}$$

2. Equilateral Configurations

In this paper, we are interested in solutions in which the three bodies are at all times at the vertices of an equilateral triangle. Of course, the triangle rotates about the center of mass of the system which we assume without loss of generality is the origin of the complex plane. Let z_j^*,

$j = 0, \ldots, 2$, denote such a solution. Let R denote the distance between any pair of these bodies. We suppose that R does not change with time. With these assumptions, we get

$$\begin{aligned} \ddot{z}_j^* &= \frac{G}{R^3} \sum_{k \neq j} m_k (z_k^* - z_j^*) \\ &= \frac{G}{R^3} \sum_{k} m_k (z_k^* - z_j^*) \\ &= \frac{G}{R^3} \left(\sum_{k} m_k z_k^* - M z_j^* \right) \\ &= -\frac{GM}{R^3} z_j^* \end{aligned} \quad (2)$$

where $M = \sum_k m_k$ denotes the total mass of the three bodies and, of course, we have used the fact that the center of mass is the origin when we replaced $\sum_k m_k z_k^* = 0$.

To determine the rate of rotation, assume that there is some common frequency ω and therefore that the positions are given by

$$z_j^* = r_j e^{i\omega t + i\theta_j} = e^{i\omega t} z_j^*(0)$$

where

$$z_j^*(0) = r_j e^{i\theta_j}$$

denotes the initial position of body j. Differentiating with respect to time twice, we get

$$\ddot{z}_j^* = -\omega^2 z_j^*.$$

Combining this with (2), we get that

$$\omega = \sqrt{\frac{GM}{R^3}}. \quad (3)$$

3. Counter-Rotated Coordinates

The equations of motion are slightly more complicated in a rotating frame. But, the ability to freeze all three bodies outweighs the additional complexity yielding an overall dramatic simplification. To this end, let

$$\xi_j = e^{-i\omega t} z_j. \quad (4)$$

Then,

$$\dot{\xi}_j = -i\omega \xi_j + e^{-i\omega t} \dot{z}_j \quad (5)$$

and

$$\ddot{\xi}_j = -i\omega\dot{\xi}_j - i\omega e^{-i\omega t}\dot{z}_j + e^{-i\omega t}\ddot{z}_j. \tag{6}$$

Using (5) to eliminate $\dot{z}_j$ and (1) to eliminate $\ddot{z}_j$, we arrive at the equations of motion in the rotating frame

$$\ddot{\xi}_j = G\sum_{k\neq j} m_k \frac{\xi_k - \xi_j}{|\xi_k - \xi_j|^3} + \omega^2\xi_j - 2i\omega\dot{\xi}_j. \tag{7}$$

Let ξ_j^* denote the counter-rotated "motion" of z^*. The beauty here is that this motion is motionless:

$$\xi_j^*(t) = e^{-i\omega t}z_j^* = z_j^*(0).$$

Our aim is to consider a general motion ξ_j, $j = 0, \ldots, 2$, that is just a small perturbation (at least in the initial conditions) from the given motionless motion ξ_j^*, $j = 0, \ldots, 2$. Before doing that, we digress briefly to discuss choices of independent variables in the complex plane.

4. Independent Variables in the Complex Plane

Forget for the moment that all of our equations are functions of time. Equation (7) can be thought of as a simple nonlinear relation

$$w = F(z)$$

where w and z are both complex. We wish to linearize F. Normally, at this stage, one resorts to real and imaginary parts and writes

$$u + iv = F(x + iy)$$

and then deduces a pair of functions g and h that represents the real and imaginary parts of F

$$u = g(x, y) \qquad \text{and} \qquad v = h(x, y).$$

One then expands g and h according to the first two terms of a Taylor series. That is the standard method. Sometimes it is convenient to use polar coordinates instead of cartesian coordinates. But, in either case, the special character of the complex plane is at this point abandoned for more traditional notations.

But, there is a more elegant way which sticks with complex notation. In the complex plane, there are two complex variables z and $\bar{z}$. In fact, the function F that we are interested in is really a function of z and $\bar{z}$

$$w = F(z, \bar{z}).$$

A function is called analytic if it is a function of just z and not $\bar{z}$. The function appearing on the right in (7) is not analytic. Anyway, viewed like this, a linearization of F can be written quite succinctly

$$w \approx F(z^*, \bar{z}^*) + \frac{\partial F}{\partial z}(z^*, \bar{z}^*)\,(z - z^*) + \frac{\partial F}{\partial \bar{z}}(z^*, \bar{z}^*)\,(\bar{z} - \bar{z}^*)\,.$$

We follow this approach in the next section.

5. Linear Stability Analysis

The variables on the right-hand side of (7) are ξ_k, $\bar{\xi}_k$, $\dot{\xi}_k$, and $\dot{\bar{\xi}}_k$ for $k = 0, \ldots, 2$. Linearizing by taking derivatives with respect to each of these variables and letting $\Delta\xi_k$ denote $\xi_k - \xi_k^*$, we get

$$\begin{aligned}
\ddot{\xi}_j \;\; \approx \;\; & \ddot{\xi}_j^* + G \sum_{k \neq j} m_k \frac{|\xi_k^* - \xi_j^*|^3 - (\xi_k^* - \xi_j^*)\frac{3}{2}|\xi_k^* - \xi_j^*|(\bar{\xi}_k^* - \bar{\xi}_j^*)}{|\xi_k^* - \xi_j^*|^6} \Delta\xi_k - G \sum_{k \neq j} m_k \frac{3}{2} \frac{(\xi_k^* - \xi_j^*)^2}{|\xi_k^* - \xi_j^*|^5} \Delta\bar{\xi}_k \\
& + G \sum_{k \neq j} m_k \frac{-|\xi_k^* - \xi_j^*|^3 + (\xi_k^* - \xi_j^*)\frac{3}{2}|\xi_k^* - \xi_j^*|(\bar{\xi}_k^* - \bar{\xi}_j^*)}{|\xi_k^* - \xi_j^*|^6} \Delta\xi_j + G \sum_{k \neq j} m_k \frac{3}{2} \frac{(\xi_k^* - \xi_j^*)^2}{|\xi_k^* - \xi_j^*|^5} \Delta\bar{\xi}_j \\
& + \omega^2 \Delta\xi_j - 2i\omega\Delta\dot{\xi}_j \\
\\
= \;\; & \ddot{\xi}_j^* - \frac{1}{2}\frac{G}{R^3} \sum_{k \neq j} m_k \Delta\xi_k - \frac{3}{2}\frac{G}{R^5} \sum_{k \neq j} m_k (\xi_k^* - \xi_j^*)^2 \Delta\bar{\xi}_k \\
& + \frac{1}{2}\frac{G}{R^3} \sum_{k \neq j} m_k \Delta\xi_j + \frac{3}{2}\frac{G}{R^5} \sum_{k \neq j} m_k (\xi_k^* - \xi_j^*)^2 \Delta\bar{\xi}_j \\
& + \omega^2 \Delta\xi_j - 2i\omega\Delta\dot{\xi}_j
\end{aligned}$$

(to compute these derivatives we have used the fact that $|z| = \sqrt{z\bar{z}}$). The expression simplifies somewhat if we introduce notation for the reduced mass

$$\mu_k = \frac{m_k}{M}$$

and if we use (2) to replace G/R^3 with ω^2/M:

$$\begin{aligned}
\Delta\ddot{\xi}_j &\approx -\frac{1}{2}\omega^2 \sum_{k\neq j} \mu_k \Delta\xi_k - \frac{3}{2}\omega^2 \sum_{k\neq j} \mu_k \frac{\xi_k^* - \xi_j^*}{\bar{\xi}_k^* - \bar{\xi}_j^*} \Delta\bar{\xi}_k \\
&\quad + \frac{1}{2}\omega^2 \sum_{k\neq j} \mu_k \Delta\xi_j + \frac{3}{2}\omega^2 \sum_{k\neq j} \mu_k \frac{\xi_k^* - \xi_j^*}{\bar{\xi}_k^* - \bar{\xi}_j^*} \Delta\bar{\xi}_j \\
&\quad + \omega^2 \Delta\xi_j - 2i\omega\Delta\dot{\xi}_j \qquad (8)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}\omega^2 \sum_{k=0}^{2} \mu_k \Delta\xi_k - \frac{3}{2}\omega^2 \sum_{k\neq j} \mu_k \frac{\xi_k^* - \xi_j^*}{\bar{\xi}_k^* - \bar{\xi}_j^*} \Delta\bar{\xi}_k \\
&\quad + \frac{3}{2}\omega^2 \sum_{k\neq j} \mu_k \frac{\xi_k^* - \xi_j^*}{\bar{\xi}_k^* - \bar{\xi}_j^*} \Delta\bar{\xi}_j \\
&\quad + \frac{3}{2}\omega^2 \Delta\xi_j - 2i\omega\Delta\dot{\xi}_j. \qquad (9)
\end{aligned}$$

6. The Restricted Planar 3-Body Problem

Suppose now that $\mu_2 = 0$. In this case, one may assume that $\Delta\xi_0 = \Delta\xi_1 = \Delta\bar{\xi}_0 = \Delta\bar{\xi}_1 \equiv 0$ and hence (9) reduces to just a single equation

$$\Delta\ddot{\xi}_2 = \frac{1}{2}\omega^2\Delta\xi_2 + \frac{3}{2}\omega^2 \sum_{k=0}^{1} \mu_k \frac{\xi_k^* - \xi_2^*}{\bar{\xi}_k^* - \bar{\xi}_2^*} \Delta\bar{\xi}_2 + \omega^2\Delta\xi_2 - 2i\omega\Delta\dot{\xi}_2$$

and, of course, its conjugate

$$\Delta\ddot{\bar{\xi}}_2 = \frac{1}{2}\omega^2\Delta\bar{\xi}_2 + \frac{3}{2}\omega^2 \sum_{k=0}^{1} \mu_k \frac{\bar{\xi}_k^* - \bar{\xi}_2^*}{\xi_k^* - \xi_2^*} \Delta\xi_2 + \omega^2\Delta\bar{\xi}_2 - 2i\omega\Delta\dot{\bar{\xi}}_2.$$

These linearized equations can be summarized as follows

$$\begin{bmatrix} \Delta\dot{\xi}_2 \\ \Delta\dot{\bar{\xi}}_2 \\ \Delta\ddot{\xi}_2 \\ \Delta\ddot{\bar{\xi}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{2}\omega^2 & \frac{3}{2}\omega^2 \sum_{k=0}^{1} \mu_k \alpha_k & -2i\omega & 0 \\ \frac{3}{2}\omega^2 \sum_{k=0}^{1} \mu_k \bar{\alpha}_k & \frac{3}{2}\omega^2 & 0 & 2i\omega \end{bmatrix} \begin{bmatrix} \Delta\xi_2 \\ \Delta\bar{\xi}_2 \\ \Delta\dot{\xi}_2 \\ \Delta\dot{\bar{\xi}}_2 \end{bmatrix}, \qquad (10)$$

where we have introduced the abbreviation

$$\alpha_k = \frac{\xi_k^* - \xi_2^*}{\bar{\xi}_k^* - \bar{\xi}_2^*}.$$

Let A denote the matrix in (10). The system satisfies linear stability if none of the eigenvalues of A are strictly in the right halfplane. It's a little bit tedious but not particularly hard to compute the determinant of $\lambda I - A$. The result is

$$\det(\lambda I - A) = \lambda^4 + \omega^2 \lambda^2 + \frac{9}{4}\omega^4 \left(1 - b^2\right) \tag{11}$$

where

$$b = \left| \sum_{k=0}^{1} \mu_k \alpha_k \right|.$$

Setting the determinant to zero and using the quadratic formula, we get that

$$\lambda^2 = \frac{\omega^2}{2}\left(-1 \pm \sqrt{-8 + 9b^2}\right).$$

In order to have all of the λ's lie in the negative halfplane (or on the imaginary axis), it is necessary and sufficient that λ^2 be real and negative. Hence, we must insist that

$$9b^2 \geq 8.$$

The final step is to evaluate b^2. From its definition, we have

$$\begin{aligned} b^2 &= \sum_{k=0}^{1}\sum_{k'=0}^{1} \mu_k \mu_{k'} \alpha_k \bar{\alpha}_{k'} \\ &= \sum_{k=0}^{1}\sum_{k'=0}^{1} \mu_k \mu_{k'} \frac{\xi_k^* - \xi_2^*}{\bar{\xi}_k^* - \bar{\xi}_2^*} \frac{\bar{\xi}_{k'}^* - \bar{\xi}_2^*}{\xi_{k'}^* - \xi_2^*} \\ &= \mu_0^2 + \mu_1^2 + \left(\frac{\xi_1^* - \xi_2^*}{\bar{\xi}_1^* - \bar{\xi}_2^*} \frac{\bar{\xi}_0^* - \bar{\xi}_2^*}{\xi_0^* - \xi_2^*} + \frac{\xi_0^* - \xi_2^*}{\bar{\xi}_0^* - \bar{\xi}_2^*} \frac{\bar{\xi}_1^* - \bar{\xi}_2^*}{\xi_1^* - \xi_2^*} \right). \end{aligned}$$

The expression in parentheses is invariant under rotations and translations of the coordinate system. Hence, we can assume that the equilateral triangle is embedded in the complex plane in any convenient way. For example, we can assume that $\xi_2^* = 0$, $\xi_1^* = R$, and $\xi_0^* = Re^{i\pi/3}$. Then it is easy to evaluate the expression. The result is

$$\left(\frac{\xi_1^* - \xi_2^*}{\bar{\xi}_1^* - \bar{\xi}_2^*} \frac{\bar{\xi}_0^* - \bar{\xi}_2^*}{\xi_0^* - \xi_2^*} + \frac{\xi_0^* - \xi_2^*}{\bar{\xi}_0^* - \bar{\xi}_2^*} \frac{\bar{\xi}_1^* - \bar{\xi}_2^*}{\xi_1^* - \xi_2^*} \right) = 1 \cdot \frac{Re^{-i\pi/3}}{Re^{i\pi/3}} + \frac{Re^{i\pi/3}}{Re^{-i\pi/3}} \cdot 1 = e^{-2\pi i/3} + e^{2\pi i/3} = 2\cos(2\pi/3) = -1.$$

Hence, the inequality for linear stability simplifies to

$$\mu_0^2 + \mu_1^2 - \mu_0 \mu_1 \geq \frac{8}{9}.$$

To finish the derivation, we now exploit the fact that $\mu_1 = 1 - \mu_0$ to rewrite the inequality with just one variable

$$27\mu_0^2 - 27\mu_0 + 1 \geq 0.$$

By the quadratic formula, this inequality is equivalent to

$$\mu_0 \leq \frac{1}{2}(1 - \sqrt{23/27}) \qquad \text{or} \qquad \mu_0 \geq \frac{1}{2}(1 - \sqrt{23/27}).$$

Since we assumed that body 0 is more massive than body 1, it follows that $\mu_0 \geq 1/2$ and so we can focus just on the second condition. And, finally, since $\mu_0 = m_0/(m_0 + m_1)$, we can rearrange that condition to

$$\frac{m_0}{m_1} \geq \frac{1 + \sqrt{23/27}}{1 - \sqrt{23/27}} = \frac{25 + \sqrt{23/27}}{2} = 24.9599.$$

7. Lagrange Points L4/L5 are Always Stable in Classical Flatland

Assume that the force due to gravity decays like one over distance, such as might happen in a non-relativistic 2-D "flatland". Then an analysis similar to that above can be easily carried out. The first minor difference is that the relation between G, M, R, and ω needs to be changed to

$$\omega = \frac{\sqrt{GM}}{R}.$$

Also, the flatland analogue of (11) is

$$\lambda^4 + 2\omega^2\lambda^2 + \omega^4(1 - b^2) = 0.$$

Solving the quadratic equation for λ^2, we get

$$\lambda^2 = \omega^2(-1 \pm b).$$

We already know that $b \geq 0$. In order to have λ^2 be a negative real, it is therefore necessary and sufficient to have $b^2 \leq 1$. The formula for b^2 remains unchanged from what it was before

$$b^2 = \mu_0^2 + \mu_1^2 - \mu_0\mu_1.$$

Using the fact that $\mu_1 = 1 - \mu_0$, it is easy to see that the condition $b^2 \leq 1$ is equivalent to $\mu_0 \leq 1$, which holds by definition. Hence, *L4 and L5 Lagrange points for the restricted planar 3-body problem in a classical flatland are stable.*

It is interesting to compare this with gravitation in relativistic flatland. Gott and Alpert (1984) showed that a mass in flatland warps space into a cone vith vertex located at the mass. Hence, a pair of masses don't feel each other unless they actually collide. There is no nontrivial 2-body problem nor is there a 3-body problem.

REFERENCES

J.R. Gott and M. Alpert. General relativity in a $(2+1)$-dimensional space-time. *General Relativy and Gravitation*, 16:243–247, 1984.

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