LAGRANGE POINTS $L_1$, $L_2$ AND $L_3$

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1. TWO BODIES

Let $R$ denote the Sun-Earth distance. Let $r$ denote the Earth-$L_2$ distance. Let $M$ denote the mass of the Sun. Let $m$ denote the mass of the Earth. Let $\rho$ denote the distance from the center of the Sun to the center of mass of the system. Let

$$z_E(t) = (R - \rho)e^{2\pi it/T}$$

denote Earth’s orbit about the center of mass of the system and let

$$z_S(t) = -\rho e^{2\pi it/T}$$

denote the orbit of the Sun about the center of mass. The fact that the center of mass is at the origin in our coordinate system leads to

$$mz_E(t) + Mz_S(t) = 0$$

which simplifies to

$$m(R - \rho) = M\rho$$

which we can solve for $\rho$:

$$\rho = \frac{m}{m + M} R$$

Earth’s orbit satisfies Newton’s law of motion:

$$mz''_E(t) = -GM(z_E - z_S)/|z_E - z_S|^3$$

Of course, we can solve for $T$ in terms of the masses and $R$ using

$$z''_E(t) = -\frac{4\pi^2}{T^2} z_E(t) = -\frac{4\pi^2}{T^2} R - \rho \left(z_E(t) - z_S(t)\right) = -\frac{4\pi^2}{T^2} \frac{M}{M + m} \left(z_E(t) - z_S(t)\right)$$

We get

$$\frac{4\pi^2}{T^2} \frac{M}{M + m} = GM/R^3.$$ 

Hence,

$$T = 2\pi \sqrt{\frac{R^3}{G(M + m)}}.$$ 

2. THIRD BODY

Now we consider a third body with infinitesimally small mass. We will call it the Lagrange body. Let

$$z_L(t) = (R - \rho + r)e^{2\pi it/T}$$

denote its orbit. If we assume that $r$ is a small positive constnat, then this new body can be thought of as orbiting at the Sun/Earth $L_2$ point. But, if $r$ is negative, this body could be viewed as orbiting at $L_1$ or even at $L_3$. And, if we let $r$ be complex valued, then this body could be at L4 or L5. The $L_2$ orbit must also satisfy Newton’s law of motion:

$$z''_L(t) = -GM(z_L - z_S)/|z_L - z_S|^3 - Gm(z_L - z_E)/|z_L - z_E|^3.$$
Again, we can simplify using
\[ z''_L(t) = -\frac{4\pi^2}{T^2} z_L(t) \]
\[ z_L - z_S = (R + r)e^{2\pi it/T} = \frac{R + r}{R - \rho + r} z_L \]
and
\[ z_L - z_E = re^{2\pi it/T} = \frac{r}{R - \rho + r} z_L. \]

We get
\[ (M + m)\frac{1}{R^3} = \frac{M}{R - \rho + r} + \frac{1}{(R + r)^2} + \frac{m}{R - \rho + r} \frac{1}{r^2}. \]

At this point things get tricky if \( r \) is not real because the length of a complex number involves the number, its conjugate and something to the 3/2’s power. So, henceforth, let’s assume that \( r \) is real. In this case, our formula simplifies to
\[ (M + m)\frac{1}{R^3} = \varepsilon_1 M + \frac{1}{R - \rho + r} + \varepsilon_2 m \frac{1}{R - \rho + r} \frac{1}{r^2} \]
where \( \varepsilon_1 = 1 \) and \( \varepsilon_2 = 1 \) when \( r \) is positive (the \( L_2 \) scenario), \( \varepsilon_1 = 1 \) and \( \varepsilon_2 = -1 \) when \( r \) is negative but \( R + r \) is positive (the \( L_1 \) scenario) and \( \varepsilon_1 = -1 \) and \( \varepsilon_2 = -1 \) when both \( r \) and \( R + r \) are negative (the \( L_3 \) scenario). Cross multiplying, we get
\[ (M + m)r^2(R - \rho + r)(R + r)^2 = \varepsilon_1 MR^3r^2 + \varepsilon_2 mR^3(R + r)^2 \]

Now, we use the fact that \( R - \rho = \frac{M}{M + m} R \) to rewrite the equation as:
\[ r^2(MR + (M + m)r)(R + r)^2 = \varepsilon_1 MR^3r^2 + \varepsilon_2 mR^3(R + r)^2 \]

Expanding the powers, we get
\[ r^2MR(R^2 + 2Rr + r^2) + r^2(M + m)r(R^2 + 2Rr + r^2) = \varepsilon_1 MR^3r^2 + \varepsilon_2 mR^3(R^2 + 2Rr + r^2) \]

Simplifying, we get
\[ r^2MR((1 - \varepsilon_1)R^2 + 2Rr + r^2) + r^2(M + m)r(R^2 + 2Rr + r^2) = \varepsilon_2 mR^3(R^2 + 2Rr + r^2) \]

Dividing by \( M \) and \( R^3 \) and letting \( \mu = m/M \) and \( x = r/R \), we get
\[ x^2 ((1 - \varepsilon_1) + 2x + x^2) + x^2(1 + \mu)x(1 + 2x + x^2) = \varepsilon_2 \mu(1 + 2x + x^2) \]

Writing it as a simple polynomial in \( x \), we get
\[ (1 + \mu)x^5 + (3 + 2\mu)x^4 + (3 + \mu)x^3 + (1 - \varepsilon_1 - \varepsilon_2 \mu)x^2 - 2\varepsilon_2 \mu x - \varepsilon_2 \mu = 0 \]

In the Earth/Sun system, \( \mu = 3.0 \times 10^{-6} \). For each of the three scenarios \( (L_1, L_2, L_3) \), we used Python to find the roots to this 5-th degree polynomial. In each case, we find that four of the roots are complex. The sole real root is root of interest. Here’s what we get:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>1 + x</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_1 )</td>
<td>0.99003345</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>1.01003322</td>
</tr>
<tr>
<td>( L_3 )</td>
<td>-0.99999825</td>
</tr>
</tbody>
</table>

For the \( L_2 \) case, our answer is in pretty close agreement to the value one finds on Wikipedia: \( 1 + x = 1.01004 \).