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STOCHASTIC WAVES[1]

BY

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ABSTRACT. Let ϕ be a real valued function defined on the state space of a Markov process x_t. Let τ_t be the first time x_t gets to a level set of ϕ which is t units higher than the one on which it started. We call the time changed process $\tilde{x}_t = x_{\tau_t}$ a stochastic wave. We give conditions under which this process is Markovian and we evaluate its infinitesimal operator.

1. Introduction.

1.1. Transforming a Markov process x_t by a random time change τ_t is an important tool in stochastic analysis. Usually, τ_t is the inverse of an additive functional of x_t. We consider a different type of random time change: let ϕ be a function defined on the state space of x_t and put

$$\tau_t = \inf\{s > 0; \phi(x_s) > \phi(x_0) + t\}. \tag{1.1}$$

In words, τ_t is the first time the process x_t gets to a level set of ϕ that is t units higher than the one on which it started. The time changed process,

$$\tilde{x}_t = x_{\tau_t},$$

is again a Markov process which we call the *stochastic wave* corresponding to x_t and ϕ. We assume that $\phi(x_t)$ is continuous in t. Hence

$$\phi(\tilde{x}_t) = \phi(\tilde{x}_0) + t. \tag{1.2}$$

Intuitively, this means that $\tilde{x}_t$ moves deterministically to successively higher level sets,

$$G_s = \{y: \phi(y) = s\}, \tag{1.3}$$

however, its position on a given level set is random.

Stochastic waves were used in **[8]** to study harmonic functions associated with several Markov processes. Such functions (first introduced in **[2]**) are related to a certain class of higher order partial differential equations which are neither elliptic nor hyperbolic. In **[8]**, a probabilistic formula for the solution of the Dirichlet problem for these equations was given. It involves stochastic waves and their infinitesimal operators. In this paper we investigate the infinitesimal operators of stochastic waves corresponding to diffusions.

1.2. One example of a stochastic wave has been known for a long time; namely, the one corresponding to Brownian motion $x_t = (x_t^1, x_t^2)$ in R^2 and the function

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$\phi(x^1, x^2) = x^1$. It was shown in [6] that $\tilde{x}_t^2$ is the symmetric Cauchy process with infinitesimal operator given by

$$(1.4) \qquad Hf(x) = \frac{1}{\pi}\int_R (f(y) - f(x) - (y-x)_c f'(x))(y-x)^{-2}\,dy,$$

where c is strictly positive and u_c means $u1_{|u|<c}$ (Hf does not depend on c). Of course $\tilde{x}_t^1$ is uniform motion and so the infinitesimal operator $\tilde{A}$ of $\tilde{x}_t = (\tilde{x}_t^1, \tilde{x}_t^2)$ acts on smooth functions by the formula

$$(1.5) \qquad \tilde{A}f(x^1, x^2) = \frac{\partial f}{\partial x^1}(x^1, x^2) + Hf(x^1, x^2),$$

where H is applied to f as a function of x^2 holding x^1 fixed. If the semigroup associated with the symmetric Cauchy process is regarded as acting on $L^2(R^2)$, then its infinitesimal operator H is minus the Hilbert transform of the derivative (see e.g. [7]).

A formula similar to (1.5) was given in [8] for the stochastic wave corresponding to Brownian motion in R^d and $\phi(x) = |x|$.

1.3. In this paper we prove that the infinitesimal operator of a stochastic wave corresponding to a diffusion and a smooth function ϕ has the form

$$(1.6) \qquad \tilde{A}f(x) = c(x)\left(\frac{df}{dn}(x) + Hf(x)\right).$$

Here $c(x) = |\nabla\phi(x)|^{-1}$, $df(x)/dn$ is the derivative of f at x in the direction of the exterior normal to the boundary

$$G^x = \{y\colon \phi(y) = \phi(x)\}$$

of the set

$$\Gamma^x = \{y\colon \phi(y) \leq \phi(x)\},$$

and $Hf(x)$ is the interior normal derivative at x of the harmonic[2] function in Γ^x which coincides with f on G^x (see Theorem 1). Roughly speaking, the first term captures the deterministic component expressed by (1.2) and the second term describes the random position of $\tilde{x}_t$ on each level set of ϕ. Theorem 2 establishes an integral formula similar to (1.4) for the operator H. This means that, if we disregard the deterministic component, the stochastic wave behaves locally at point x as an infinitely divisible process with Levy-Khintchine measure equal to the normal derivative of the harmonic measure for Γ^x.

1.4. In addition to the applications considered in [8], stochastic waves are useful for the study of Gaussian random fields associated with Markov processes.

1.5. In §2 we first make some definitions and establish notations so that, by the end of the section, we are able to give precise formulations of the results outlined above. §§3 and 4 are devoted to the proofs of the main results.

[2]We say that a function h is *harmonic* in a set Γ if $Ah = 0$ in the interior of Γ, where A is the differential generator of the diffusion.

2. The main results.

2.1. We use the terminology and notations common in the theory of Markov processes (see e.g. [1]). Let ϕ be a measurable function on the state space E of a strong Markov process[3] $X = (x_t, \mathfrak{F}_t, P_x, \theta_t)$. Let τ_t be the random time change defined by (1.1) and put $\tilde{\mathfrak{F}}_t = \mathfrak{F}_{\tau_t}$ and $\tilde{\theta}_t = \theta_{\tau_t}$. The process $\tilde{X} = (\tilde{x}_t, \tilde{\mathfrak{F}}_t, P_x, \tilde{\theta}_t)$ is called *the stochastic wave corresponding to X and* ϕ. To insure that $\tilde{X}$ is a Markov process, it is sufficient that the following conditions hold P_x-almost surely for all $x \in E$:

2.1.A. $\tau_t < \infty$ for all $t > 0$.

2.1.B. $\phi(x_t)$ is continuous in t.

2.1.C. $\tau_0 = 0$.

Indeed, condition 2.1.A guarantees that $\tilde{x}_t$ is defined for all $t > 0$, condition 2.1.B implies that

$$\theta_{\tau_s}\tau_t = \tau_{t+s} - \tau_s,$$

and this, together with 2.1.C, implies that $\tilde{X}$ is a strong Markov process.

2.2. Let $\mathfrak{B}$ denote the class of bounded measurable functions on E. For functions f and f_t, $t > 0$, in $\mathfrak{B}$, we write $f = s\text{-}\lim_{t\downarrow 0} f_t$ if f_t converges uniformly to f as $t \downarrow 0$. The semigroup T_t and infinitesimal operator A are defined by the formulas

$$T_t f(x) = P_x f(x_t), \qquad f \in \mathfrak{B},$$
$$Af = s\text{-}\lim_{t\downarrow 0}(T_t f - f)/t, \qquad f \in \mathfrak{D},$$

where $\mathfrak{D}$ consists of those functions $f \in \mathfrak{B}$ for which this limit exists.

The analogs for $\tilde{X}$ will be denoted $\tilde{T}_t$, $\tilde{A}$, $\tilde{\mathfrak{D}}$.

We consider the topology $\mathcal{C}$ in E generated by sets $\{x\colon r < \phi(x) < s\}$ (in other words, the weakest topology such that ϕ is continuous). We claim that the operator $\tilde{A}$ is local in $\mathcal{C}$, i.e., for every $x \in E$ and every $\mathcal{C}$-neighborhood U of x, $\tilde{A}f_1(x) = \tilde{A}f_2(x)$ if $f_1, f_2 \in \tilde{\mathfrak{D}}$ and $f_1 = f_2$ in U. Indeed, let I be an open interval which contains $\phi(x)$. Then, for sufficiently small t, $\phi(x) + t \in I$ and, by (1.2), $\tilde{x}_t \in \phi^{-1}(I) = U$ a.s. P_x. Therefore, $\tilde{T}_t f(x)$ depends only on values of f in U and so does $\tilde{A}f(x)$.

We put $f \in \tilde{\mathfrak{D}}_x$ if there exist $f^* \in \tilde{\mathfrak{D}}$ and a $\mathcal{C}$-neighborhood U of x such that $f = f^*$ in U. For every $f \in \tilde{\mathfrak{D}}_x$ we put $\tilde{A}f(x) = \tilde{A}f^*(x)$ (the right side does not depend on U and f^*).

2.3. Let $\mathbf{U}$ denote the collection of unit vectors in m-dimensional Euclidean space R^m. For a differentiable function f and a vector $u \in \mathbf{U}$, let

$$\frac{df}{du}(y) = \nabla f(y)\cdot u$$

denote the derivative of f in the direction of u.

For every set Γ in R^m having nonempty interior Γ^0, let $C^{l,\lambda}(\Gamma)$, $l \geq 0$, $0 \leq \lambda \leq 1$, denote the class of functions f which have derivatives up to order l in Γ^0 with

[3]We assume throughout that the killing time ζ is identically infinity.

continuous extensions to Γ and for which

$$(2.1)\qquad |f|_{l,\lambda;\Gamma} = \sum_{j\leq l} \sup \left| \frac{d^j f}{du_1 \cdots du_j}(y) \right| + \sup \overline{\lim_{z\to y}} \frac{\left| \frac{d^l f}{du_1 \cdots du_l}(y) - \frac{d^l f}{du_1 \cdots du_l}(z) \right|}{|y-z|^\lambda}$$

is finite[4] (the suprema are taken over all vectors $u_i \in U(y)$ and all $y \in \Gamma$).

A Markov process X in R^m is called a *diffusion* if $\mathfrak{D}$ contains $C^{2,0}(R^m)$ and if the infinitesimal operator A restricted to functions $f \in C^{2,0}(R^m)$ is a second order elliptic differential operator

$$Af(x) = \sum_{ij} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} f(x) + \sum_i b^i(x) \frac{\partial}{\partial x^i} f(x)$$

whose coefficients a^{ij} and b^i are of class $C^{0,\lambda}(R^m)$ for some $\lambda > 0$. The operator A when restricted to $C^{2,0}(R^m)$ is called the *differential generator* of X.

The diffusion with coefficients $a^{ij} = \delta_{ij}$ and $b^i = 0$ is called *Brownian motion*.

2.4. We denote by $\tau(\Gamma)$ the first exit time of x_t from Γ

$$\tau(\Gamma) = \inf\{t\colon t > 0, x_t \notin \Gamma\}.$$

Let $\Gamma_s = \{y\colon \phi(y) \leq s\}$ (cf (1.3)).

THEOREM 1. *Let X be a diffusion in R^m and let ϕ be a continuous function. Suppose that*

2.4.A. $\tau(\Gamma_s) < \infty$ *a.s. P_x for all $x \in R^m$, $s \in R$.*

2.4.B. *Each point of G_s is a regular point*[5] *for Γ_s^c.*

Then $\tilde{X}$ is a strong Markov process.

Suppose, in addition, that, for some $x \in R^m$, Γ^x is bounded and there exists a neighborhood V of G^x such that

2.4.C. ϕ *is of class $C^{2,\lambda}(\overline{V})$,*

2.4.D. $\nabla\phi \neq 0$ *in $\overline{V}$.*

Then every function f in $C^{2,\lambda}(V)$ is in $\tilde{\mathfrak{D}}_x$ and $Af(x)$ is given by (1.6).[6]

2.5. Let Γ be a closed bounded domain in R^m with a smooth boundary $\partial\Gamma$. Then there exists a unique continuous function $p(y, z)$, $y \in \Gamma$, $z \in \partial\Gamma$, $y \neq z$, such that, for all $f \in \mathfrak{B}$,

$$P_y f(x_{\tau(\Gamma)}) = \int_{\partial\Gamma} p(y, z) f(z) s(dz),$$

where $\tau(\Gamma)$ is the first exit time from Γ and s is surface area on $\partial\Gamma$ (see [**5**, §21 and **1**, §2, Chapter 13]). We call *p the Poisson kernel for* Γ ($p(y, z)s(dz)$ is *the harmonic measure for* Γ).

[4] If $\lambda = 0$, we omit the second sum.

[5] x is a regular point for Γ^c if $P_x\{\tau(\Gamma) = 0\} = 1$.

[6] The case of unbounded Γ^x is not covered by Theorem 1. It requires additional consideration.

THEOREM 2. *Suppose that X is Brownian motion in R^m. $m \geqslant 2$. Let ϕ and x satisfy the conditions of Theorem 1. Then the Poisson kernel p for Γ^x has the following properties*:

2.5.A. *For every boundary point $z \neq x$, p has an interior normal derivative with respect to the first variable*: $-dp(x, z)/dn_x$.

2.5.B. *There is a constant K, such that $|dp(x, z)/dn_x| \leqslant K|x - z|^{-m}$, for all $z \in \partial\Gamma^x$.*

For $f \in C^{2,\lambda}(V)$,

$$(2.2) \qquad Hf(x) = \int_{\partial\Gamma^x} \frac{dp}{dn_x}(x, z)(f(x) - f(z) - \nabla_{\mathcal{P}} f(x)\cdot (x - z))s(dz),$$

where $\nabla_{\mathcal{P}} f(x)$ is the projection of the gradient $\nabla f(x)$ onto the tangent plane to $\partial\Gamma^x$ at x.

3. Proof of Theorem 1.

3.1. Let V be a neighborhood of G^x described in Theorem 1. There exists an interval $I = (a_0, b_0)$ containing $\phi(x)$ such that Γ_s is bounded and $\partial\Gamma_s = G_s$ for all $s \in [a_0, b_0]$.

For $y \in V$, put $\mathbf{U}(y) = \{u \in \mathbf{U}\colon u \cdot \nabla\phi(y) = 0\}$. For $s \in I$ and $v \in C^{2,\lambda}(\Gamma_s)$, we define a boundary norm $|v|_{l,\lambda;G_s}$ by (2.1) where the suprema are taken over all $u_i \in \mathbf{U}(y)$ and all $y \in G_s$. We will need the following Schauder estimate: there is a constant K such that for any $s \in I_0$ and any $v \in C^{2,\lambda}(\Gamma_s)$,

$$(3.1) \qquad |v|_{2,\lambda;\Gamma_s} \leqslant K\{|Av|_{0,\lambda;\Gamma_s} + |v|_{2,\lambda;g_s}\}.$$

This is proved for example in [**4**, §3.2].

3.2. It follows from conditions 2.4.A and B and the continuity of trajectories of diffusions that $\tilde{X}$ satisfies 2.1.A, B and C and so it is a strong Markov process.

Let $a_0 < a_1 < \phi(x) < b_1 < b_0$. Put $V_i = \{a_i \leqslant \phi \leqslant b_i\}$, $i = 0, 1$. For every $f \in C^{2,\lambda}(V)$, there exists a function of class $C^{2,\lambda}(R^m)$ which vanishes outside V_1 and coincides with f in a neighborhood of the level set G^x. Therefore we can assume without loss of generality that f vanishes on the complement of V_1.

There exists $\delta > 0$ such that, for $0 < t < \delta$, $G_{s+t} \subset V_1^c$ if $s \notin I$, and $G_{s+t} \subset V$ if $s \in I$. For $s \in I$ put

$$(3.2) \qquad v_s(y) = P_y f(x_{\tau(\Gamma_s)}) - f(y).$$

By Theorem 13.9 in [**1**], v_s is the unique solution of the boundary value problem

$$(3.3) \qquad Av = -Af \quad \text{in } \Gamma_s - G_s, \qquad v = 0 \quad \text{on } G_s.$$

By Theorem 36.V in [**5**], $v_s \in C^{2,\lambda}(\Gamma_s)$.

For each $y \in V_0$, let $l_y(s)$, $s \in I$, be the solution of the equation $dl/ds = \nabla\phi(l)/|\nabla\phi(l)|^2$ satisfying $l(\phi(y)) = y$. In other words, $l_y(\cdot)$ is the gradient path of ϕ which passes through y and is parametrized by the condition $l_y(s) \in G_s$ for all $s \in I$.

Let $d(\eta, \zeta)$ be the arc length of $l_z(\cdot)$ between $\eta = l_y(r)$ and $\zeta = l_y(s)$. For all $y \in V_0$, $r < s \in I$,

$$(3.4) \qquad d(\eta, \zeta) = \int_r^s |l'(w)|dw \leqslant K(s - r).$$

Put

$$\begin{aligned} n_y &= \nabla\phi(y)/|\nabla\phi(y)|, && y \in V_0,\\ Q(s, y) &= \nabla v_s(y)\cdot n_y, && s \in I, y \in \Gamma_s\\ h_t(y) &= \frac{1}{t}(\tilde{T}_t f(y) - f(y)), && t > 0, y \in R^m,\\ h(y) &= \begin{cases} -Q(\phi(y), y)|\nabla\phi(y)|^{-1}, & y \in V_0,\\ 0, & y \in V_0^c. \end{cases} \end{aligned}$$

It follows from (3.2) that $h(x)$ is equal to the right side of (1.6). Therefore Theorem 1 will be proved if we show that $h = s\text{-}\lim_{t\downarrow 0} h_t$.

3.3. Suppose that this is not true. Then there exists a sequence t_n tending to zero, a sequence $y_n \in R^m$, and an $\varepsilon > 0$ such that $|h_{t_n}(y_n) - h(y_n)| > \varepsilon$ for all n. We may assume that $t_n < \delta$ for all n. It is easy to see that $h_t = 0$ on V_0^c for $t < \delta$ and, since $h = 0$ on $V_0^c, y_n \in V_0$.

Put $r_n = \phi(y_n)$, $s_n = \phi(y_n) + t_n$ and $z_n = l_{y_n}(s_n)$. By Cauchy's mean value theorem,

$$h_{t_n}(y_n) = -\frac{v_{s_n}(z_n) - v_{s_n}(y_n)}{\phi(z_n) - \phi(y_n)} = -|Q(s_n, \hat{y}_n)|\,|\nabla\phi(\hat{y}_n)|^{-1},$$

where $\hat{y}_n$ is a point on l_{y_n} lying between y_n and z_n. Hence,

$$|h_{t_n}(y_n) - h(y_n)| \leq |Q(s_n, \hat{y}_n) - Q(r_n, y_n)| c(\hat{y}_n) + |Q(r_n, y_n)|\,|c(\hat{y}_n) - c(y_n)|,$$

where $c = |\nabla\phi|^{-1}$. The points y_n and $\hat{y}_n$ belong to V and $d(y_n, \hat{y}_n) \to 0$. Hence by 2.4.C and D, $c(\hat{y}_n)$ is bounded and $|c(\hat{y}_n) - c(y_n)|$ tends to zero as $n \to \infty$. By (3.1) and (3.3), $|Q(r_n, y_n)| \leq |v_{r_n}|_{2,\lambda;\Gamma_{r_n}} \leq K|Af|_{0,\lambda;\Gamma_{r_n}} \leq K|Af|_{0,\lambda;V_0}$. To arrive at a contradiction we shall show that $|Q(s_n, \hat{y}_n) - Q(r_n, y_n)|$ tends to zero as $n \to \infty$.

To this end we prove that

$$|Q(s, z) - Q(r, y)| \leq K_1\left\{|s - r|^\lambda + |z - y| + |n_z - n_y|\right\}, \tag{3.5}$$

for all r, s, y and z such that $y \in V_0$ and $r = \phi(y) \leq \phi(z) \leq s \leq b + \delta$. First we write

$$|Q(s, z) - Q(r, y)| \leq \Delta_1 + \Delta_2 + \Delta_3,$$

where

$$\Delta_1 = |(\nabla v_s(z) - \nabla v_s(y))\cdot n_z|, \qquad \Delta_2 = |\nabla v_s(y)\cdot (n_z - n_y)|,$$
$$\Delta_3 = |\nabla(v_s - v_r)(y)\cdot n_y|.$$

By (3.1), we have

$$|v_s|_{2,\lambda;\Gamma_s} \leq K_2|Af|_{0,\lambda;V}. \tag{3.6}$$

By the mean value property, $\Delta_1 \leq |v_s|_{2,0;\Gamma_s}|z - y|$ and so, by (3.6), Δ_1 is bounded by a constant times $|z - y|$. Since $\Delta_2 \leq |v_s|_{1,0;\Gamma_s}|n_z - n_y|$, it follows from (3.6) that Δ_2 is bounded by a constant times $|n_z - n_y|$. To estimate Δ_3, put $w_{r,s} = v_s - v_r$. Then $w_{r,s} \in C^{2,\lambda}(\Gamma_r)$, $Aw_{r,s} = 0$ in $\Gamma_r - G_r$, and $w_{r,s} = v_s$ on G_r. Hence Δ_3 is just the magnitude of the normal derivative of a harmonic function which, according to Theorem 35,III in [5], can be estimated in terms of two derivatives of its boundary values,

(3.7) $$\Delta_3 \leqslant K_3 |v_s|_{2,0;\partial\Gamma_r}.$$

Let η be a point on $\partial\Gamma_r$ and let $\zeta = l_\eta(s)$. Applying the mean value theorem to $\psi(u) = v_s(l_\eta(u))$ and using the fact that $\psi(r) = v_s(\eta)$ and $\psi(s) = 0$ we get

(3.8) $$|v_s(\eta)| \leqslant |v_s|_{1,0;\Gamma_s} d(\eta, \zeta).$$

For any unit vector u,

(3.9) $$\left|\frac{dv_s}{du}(\eta)\right| \leqslant \left|\frac{dv_s}{du}(\eta) - \frac{dv_s}{du}(\zeta)\right| + \left|\frac{dv_s}{du}(\zeta)\right|$$

and, by the mean value theorem,

(3.10) $$\left|\frac{dv_s}{du}(\eta) - \frac{dv_s}{du}(\zeta)\right| \leqslant |v_s|_{2,0;\Gamma_s} d(\eta, \zeta).$$

To estimate the second term in (3.9), we use the fact that $v_s = 0$ on G_s to write

$$\frac{dv_s}{du}(\zeta) = (u \cdot n_\zeta)(\nabla v_s(\zeta) \cdot n_\zeta)$$

and so, for $u \in \mathbf{U}(\eta)$, we have

(3.11) $$\left|\frac{dv_s}{du}(\zeta)\right| \leqslant |v_s|_{1,0;\Gamma_s} |n_\zeta - n_\eta|.$$

Since $|\nabla\phi|^{-1}$ is bounded on $\bar{U}$, we see that $|dn_{l_{\eta(w)}}/dw|$ is bounded and so

(3.12) $$|n_\zeta - n_\eta| \leqslant \int_r^s \left|\frac{d}{dw} n_{l_{\eta(w)}}\right| dw \leqslant K_4(s - r).$$

Combining (3.4), (3.9), (3.10), (3.11) and (3.12), we find that, for $u \in \mathbf{U}(\eta)$, $|dv_s(\eta)/du|$ is bounded by a constant times $s - r$.

Now consider two vectors, $u_1, u_2 \in \mathbf{U}(\eta)$. The procedure for estimating $(d/du_1)dv_s(\eta)/du_2$ is analogous to the first derivative case except, instead of the mean value theorem, we use Hölder continuity to get

$$\left|\frac{d}{du_1}\frac{d}{du_2}v_s(\eta) - \frac{d}{du_1}\frac{d}{du_2}v_s(\zeta)\right| \leqslant |v_s|_{2,\lambda;\Gamma_s}\{d(\zeta, \eta)\}^\lambda.$$

We find then that $|(d/du_1)dv_s(\eta)/du_2|$ is bounded by a constant times $(s - r)^\lambda$ and so $|v_s|_{2,0;2\Gamma_r} \leqslant K_5(s - r)^\lambda$. Putting this into (3.7), we get (3.5).

4. Proof of Theorem 2.

4.1. Put $r = \phi(x)$ and write $\Gamma = \Gamma_r$ and $G = G_r$. Let

$$\Pi f(y) = P_y f(x_{\tau(\Gamma)}).$$

By Theorem 21,VI in [5] the Poisson kernel p for Γ exists and is given by the formula[7]

(4.1) $$p(y, z) = 2\xi(y, z) - 4\int \xi(y, z')\gamma(z', z)s(dz'), \qquad y \in \Gamma, z \in G, y \neq z,$$

[7]Formula (4.1) follows from formulas (21.2), (21.3), (17.3) and (17.8) in [5].

where ξ is the inward normal derivative with respect to the z variable of

$$g(y,z)=\begin{cases}\dfrac{1}{(m-2)\omega_m}|y-z|^{2-m}, & m\geqslant 3,\\ -\dfrac{1}{2\pi}\log|y-z|, & m=2,\end{cases}$$

and γ is defined by the integral equation

$$\gamma(y,z)=\xi(y,z)-2\int_G \xi(y,z')\gamma(z',z)s(dz'),\qquad y,z\in G, y\neq z \tag{4.2}$$

(ω_m is the surface area of the unit ball in R^m).

4.2. By iterating formula (4.2) and using the fact that convolutions of the kernel ξ with itself have lower order singularities, it is possible to show that $\gamma(y,z)=O(|y-z|^{2-m})$ and, for $y\neq z$, $\gamma(y,z)$ is twice continuously differentiable with respect to the x variables (of course, derivatives are taken in directions tangent to G). Consequently, using continuity and differentiability properties of double layer potentials (see e.g. [**5**, §15 or **3**, §VI.6]), it can be shown that p has the following properties:

4.2.A. The derivative $dp(y,z)/dn_y$ exists and can be continued to a continuous function of y in $\Gamma\setminus\{z\}$.

4.2.B. There exists a constant K such that $|dp(y,z)/dn_y|\leqslant K|x-z|^{-m}$, for all $y\in l_x\cap(\Gamma\setminus G)$, $z\in G$.

Properties 2.5.A and B follow from 3.3.A and B.

By Theorem 36,I in [**5**], $\Pi f\in C^{2,\lambda}(\Gamma)$ and so $Hf(x)=d\Pi f(x)/dn_x$ exists. It is given by

$$Hf(x)=\lim_{y\to x}\frac{\Pi f(y)-f(x)}{d(y,x)}$$

where the limit is taken as y tends to x along $l_x\cap(\Gamma\setminus G)$. Using 4.2.A and B and the fact that linear functions are harmonic, we have

$$\frac{\Pi f(y)-f(x)}{d(y,x)}=\int_G\frac{d}{dn_{\hat{y}}}p(\hat{y},z)h(z)s(dz)+\nabla_{\mathcal{G}}f(x)\cdot\frac{y-x}{d(y,x)}, \tag{4.3}$$

where $h(z)=f(x)-f(z)-\nabla_{\mathcal{G}}f(x)\cdot(x-z)$ and $\hat{y}$ is a point of l_x lying between y and x. Since $f\in C^{2,\lambda}(V)$, the function $h(z)/|x-z|^2$ is bounded on G. In view of 4.2.A and B, the integral in (4.3) tends to the one in (2.2) as y tends to x along l_x. Since $(y-x)/d(y,x)$ tends to $-n_x$ as y tends to x along l_x, we see that the second term in (4.3) tends to zero.

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