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Toward a Stochastic Calculus for Several Markov Processes

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In this paper we investigate a class of harmonic functions associated with a pair $x_t = (x^1_{t^1}, x^2_{t^2})$ of strong Markov processes. In the case where both processes are Brownian motions, a smooth function f is harmonic if

$$\Delta_{x^1}\Delta_{x^2} f(x^1, x^2) = 0.$$

For these harmonic functions we investigate a certain boundary value problem which is analogous to the Dirichlet problem associated with a single process. One basic tool for this study is a generalization of Dynkin's formula, which can be thought of as a kind of stochastic Green's formula. Another important tool is the use of Markov processes $\tilde{x}^i_{t^i}$ obtained from $x^i_{t^i}$ by certain random time changes. We call such a process a stochastic wave since it propogates deterministically through a certain family of sets; however its position on a given set is random.

1. INTRODUCTION

1.1. A stochastic calculus, developed in the theory of Markov processes, allows one to get not only an intuitive picture, but also rigorous proofs of many results in classical analysis. Recently, attempts have been made to extend this calculus by using several processes considered at different times. As early as 1969, Walsh [7] used several Brownian motions to investigate multiply harmonic functions. In [3] Dynkin introduced a class of harmonic functions associated with a family X of Markov processes. We call these functions X-harmonic (see Section 1.3 for the definition). This class is wider than the class of multiply harmonic functions.

In [3], Dynkin considered only time-reversible Markov processes and harmonic functions belonging to the Dirichlet space associated with a family of such processes. For certain domains B, it has been proved that there exists one and only one "regular" function F harmonic in B and equal quasi-everywhere on the complement B^c of B to a given function f. The

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function F has been expressed by a formula which involves values of a certain function ψ (related to f) on the entire state space.

In this paper, X-harmonic functions are studied by different methods which are applicable to all strong Markov processes. Using stochastic waves (cf. [4]) we give a probabilistic formula for F in terms of the values of f on B^c.

In the case where both formulas are applicable, one expression can be derived from the other using a stochastic analogue of Green's formula (cf. Theorems 1.1 and 3.1) which is known in the case of one process as Dynkin's formula.

For notational convenience we consider only the case of two processes but the methods are applicable to the general case as well.

1.2. We start by discussing the classical Dirichlet problem. Let B be a bounded domain in $\mathbb{R}^d$ with a smooth boundary and let f be a continuous function on $\mathbb{R}^d$. A function F is a *solution of the classical Dirichlet problem* if F is continuous, harmonic in B, and equal to f on B^c. This problem has a unique solution which can be described in terms of the Brownian motion $X = (x_t, P_x)$ by the formula[1]

$$F(x) = P_x f(x_{\tau(B)}), \tag{1.1}$$

where $\tau(B) = \inf\{t > 0: x_t \notin B\}$ is the first exit time from B (see, e.g., [5, Chap. 2]).

The condition that F is harmonic in B is equivalent to[2]:

1.2.A. *For every stopping time* $\tau \leqslant \tau(B)$, $F(x) = P_x F(x_\tau)$.

For a general strong Markov process, 1.2.A serves as the definition of an *X-harmonic* function.

Uniqueness of the solution follows immediately from Dynkin's formula: for $F \in C(\bar{B}) \cap C^2(B)$,

$$P_x F(x_{\tau(B)}) - F(x) = P_x \int_0^{\tau(B)} \tfrac{1}{2}\Delta F(x_s)\, ds. \tag{1.2}$$

Any solution must be in $C(\bar{B}) \cap C^2(B)$, and so by (1.2) it is given by (1.1). Details may be found in [5] Chap. 2.

1.3. Now consider the random field $x_t = (x^1_{t^1}, x^2_{t^2})$ corresponding to a pair of independent strong Markov processes. Time takes values in the set $\mathbf{T} = [0, \infty)^2$. This set is given the usual coordinate partial order: $t \leqslant s$ means $t^i \leqslant s^i$ for $i = 1, 2$, and $t < s$ means $t^i < s^i$ for $i = 1, 2$.

Suppose τ^i is a stopping time for the process $x^i_{t^i}$. The pair $\tau = (\tau^1, \tau^2)$ is called a *stopping vector*. An example of a stopping vector is given by

[1] If P is a measure and Y is a function, then PY means the integral of Y with respect to P.
[2] If Y and Z are functions of paths, then $Y \leqslant Z$ means $Y \leqslant Z$ a.s. P_x for all x.

$\tau(B) = (\tau^1(B_1), \tau^2(B_2))$, where $\tau^i(B_i)$ is the first exit time of $x^i_{t^i}$ from B_i. With this definition of $\tau(B)$, formula (1.1) still makes sense. In terms of the set $B = (B_1^c \times B_2^c)^c$, the function F defined by (1.1) satisfies the following mean value property:

1.3.A. *For every stopping vector τ for which $x_t \in B$ for $0 \neq t \leqslant \tau$,*

$$P_x\{F(x_\tau) - F(x_{\tau^1,0}) - F(x_{0,\tau^2}) + F(x_0)\} = 0, \text{ for all } x.$$

For any set B, we say that a function F is *X-harmonic in B* if 1.3.A holds. Our aim is to investigate a certain boundary value problem associated with X-harmonic functions.

1.4. If $x^i_{\tau^i}$ is uniform motion, B is an open set in $\mathbb{R}^2$, and $F \in C^2(B)$, then F is X-harmonic in B if and only if

$$\frac{\partial^2}{\partial x^1\,\partial x^2} F(x^1, x^2) = 0, \qquad x \in B. \tag{1.3}$$

In the coordinates $z = x^2 - x^1$, $s = x^1 + x^2$, Eq. (1.3) becomes the one-dimensional wave equation:

$$\frac{\partial^2 F}{\partial s^2} = \frac{\partial^2 F}{\partial z^2}. \tag{1.4}$$

The problem usually studied in connection with the wave equation is the Cauchy problem (see, e.g., [6] Section 1.4). This problem is well posed only in domains having noncharacteristic boundary curves. For Eq. (1.3) this means the boundary ∂B of B must be the graph of a continuous strictly monotone function. Such domains can be described by

$$B = \{x: \Phi(x) < 0\}, \tag{1.5}$$

where

$$\Phi(x) = \varphi^1(x^1) + \varphi^2(x^2), \tag{1.6}$$

and the φ^i are continuous, strictly monotone, and bounded below. For an arbitrary strong Markov process $X = (x_t, P_x)$ and a function φ on its state space, we replace the continuity and monotonicity conditions by

1.4.A. *$\varphi(x_t)$ is continuous in t a.s. P_x for all x.*
1.4.B. *$\inf\{s > 0: \varphi(x_s) - \varphi(x_0) > 0\} = 0$ a.s. P_x for all x.*

Obviously, 1.4.A, B are equivalent to continuity and monotonicity if X is uniform motion. We call a set B of the form (1.5) an *admissible domain*[3] if Φ

[3]If X^i is a diffusion, then to satisfy 1.4.A, B, it suffices that φ^i be a bounded continuous function with smooth level sets. In general, however, conditions 1.4.A, B severely limit the possibility for X^i to have jumps.

satisfies (1.6) with functions φ^i which satisfy 1.4.A, B and are bounded below.

1.5. If X^i is Brownian motion in $\mathbf{R}^{d_i}$, B is a bounded domain in $\mathbf{R}^d$ ($d = d_1 + d_2$), and $F \in C^4(B)$, then it is possible to show[4] that F is X-harmonic in B if and only if

$$\Delta_{x^1}\Delta_{x^2}F(x^1, x^2) = 0, \qquad x \in B. \tag{1.7}$$

This equation is not elliptic, parabolic, or hyperbolic. Even though $\Delta_{x^1}\Delta_{x^2}$ is the product of two elliptic operators, we will see that in some respects it behaves like a hyperbolic operator.

In the classical case, harmonicity of F actually implies that F is infinitely differentiable. In contrast, condition 1.3.A does not imply any smoothness. To see this, note that any function F of the form $F(x) = F^1(x^1) + F^2(x^2)$ satisfies 1.3.A.

1.6. To investigate X-harmonic functions we use stochastic waves and partial infinitesimal operators of certain two-parameter semigroups.

Let x_t be a strong Markov process and let φ be a measurable function which is bounded below and which satisfies 1.4.A, B. Put

$$\sigma_t = \inf\{s > 0: \varphi(x_s) - \varphi(x_0) > t\}. \tag{1.8}$$

In words, σ_t is the first time x_t gets to the level set of φ, which is t units higher than the level set at which it started. The *stochastic wave* $\tilde{X}$ corresponding to X and φ is the strong Markov process

$$\tilde{x}_t = x_{\sigma_t}, \tag{1.9}$$

obtained from X by the random time change (1.8).

It follows from 1.4.A that, for $t \geqslant 0$,

$$\varphi(\tilde{x}_t) = \varphi(\tilde{x}_0) + t. \tag{1.10}$$

Intuitively, this means that the wave moves deterministically to successively higher level sets of φ. However, its position on a given level set is random.

If X is uniform motion to the right with unit velocity, then the only functions φ which satisfy 1.4.A, B are continuous strictly increasing functions. Hence, in this case, the only stochastic waves are deterministic motions:

$$\tilde{x}_t = \varphi^{-1}(\varphi(\tilde{x}_0) + t).$$

[4]For functions $F \in C^4(B)$, we can apply Dynkin's formula twice to get

$$P_x(F(x_\tau) - F(x_{\tau^1,0}) - F(x_{0,\tau^2}) + F(x_0)) = P_x\int_0^{\tau^1}\int_0^{\tau^2}\tfrac{1}{4}\Delta_{x^1}\Delta_{x^2}F(x_t)\,dt.$$

From this identity it is easy to see that X-harmonicity is equivalent to (1.7).

To study X-harmonic functions in an admissible domain $B = \{x: \varphi^1(x^1) + \varphi^2(x^2) < 0\}$, we use two stochastic waves: $\tilde{X}^i$ ($i = 1,2$) corresponding to X^i and φ^i.

1.7. To every pair of independent Markov processes $x_t = (x^1_{t^1}, x^2_{t^2})$ there corresponds a *two-parameter semigroup*:

$$T_t f(x) = P_x f(x_t), \qquad t \in \mathbf{T}. \tag{1.11}$$

We say that f is *T_t-continuous* if $T_t f(x)$ converges boundedly[5] to $f(x)$ as $t \searrow 0$. Put

$$T^1_{t^1} = T_{t^1,0}, \qquad T^2_{t^2} = T_{0,t^2}. \tag{1.12}$$

The *weak i^{th} partial infinitesimal operator*[6] is defined by

$$A^i f = w - \lim_{t^i \searrow 0} \frac{T^i_{t^i} f - f}{t^i}. \tag{1.13}$$

The domain D^i of this operator consists of all T_t-continuous functions f for which $A^i f$ is T_t-continuous.

A function f is called *X-proper* if it is in the domain $D^{1,2}$ of the "mixed" operator: $A^{1,2} = A^1 A^2$. That is, $f \in D^1 \cap D^2$, $A^1 f \in D^2$, and $A^2 f \in D^1$. For X-proper functions f, the operators commute: $A^1 A^2 f = A^2 A^1 f$.

All constructions introduced for $X = (X^1, X^2)$ are applicable to $X = (\tilde{X}^1, \tilde{X}^2)$, and we put tildes over the corresponding symbols (e.g., $\tilde{A}^i$ is the weak i^{th} partial infinitesimal operator for $\tilde{X}$ and its domain is $\tilde{D}^i$).

1.8. The set of t for which $x_t \in B$ is hard to describe. However, the analogous set for $\tilde{x}_t$ is simply a triangle. Put

$$R(x) = \{t \in \mathbf{T}: t^1 + t^2 < \Phi_-(x)\}, \tag{1.14}$$

where Φ_- denotes the negative part[7] of Φ. From (1.5) and (1.10),

$$\{t: \tilde{x}_t \in B\} = R(\tilde{x}_0).$$

Let

$$L_x = \{t \in \mathbf{T}: t^1 + t^2 = \Phi_-(x)\}. \tag{1.15}$$

[5] We say that a family $(f_t(x))_{t>0}$ converges boundedly to $f(x)$ if $f_t(x)$ converges pointwise to $f(x)$ as $t \searrow 0$ and $\sup_{0<t<\epsilon} \sup_x |f_t(x)| < \infty$ for some $\epsilon = (\epsilon_1, \epsilon_2) > 0$. We write $f = w - \lim_{t\searrow 0} f_t$ or $f_t \xrightarrow{w} f$ as $t \searrow 0$.

[6] If X^i is uniform motion to the right, then for functions f which have a derivative with respect to x^i, $A^i f = \partial f/\partial x_i$.

[7] For every real number a, we denote by a_+ the positive part and by a_- the negative part of a; i.e., $a_+ = \max(a,0)$ and $a_- = \max(-a,0)$.

$R(x)$ is a triangle with vertices $0, (\Phi_-(x), 0)$ and $(0, \Phi_-(x))$. L_x is the side opposite 0. One of our main results is the following theorem.

THEOREM 1.1. *For every $\tilde{X}$-proper function f,*

$$f(x) = P_x\left\{\frac{1}{2}f\left(\tilde{x}_{\Phi_-(x),0}\right) + \frac{1}{2}f\left(\tilde{x}_{0,\Phi_-(x)}\right) - \int_{L_x} \tilde{A}_N f(\tilde{x}_t)\, dt^1 \right.$$
$$\left. + \int_{R(x)} \tilde{A}^{1,2} f(\tilde{x}_t)\, dt\right\}, \tag{1.16}$$

where

$$\tilde{A}_N f = \tfrac{1}{2}(\tilde{A}^1 f + \tilde{A}^2 f). \tag{1.17}$$

Theorem 1.1 is a particular case of Theorem 3.1 which is proved in Section 3.1.

For $x \in B$, (1.16) gives the value of f at x in terms of the values of f and $\tilde{A}_N f$ on B^c (actually on $\{y: \Phi(y) = 0\}$), and values of $\tilde{A}^{1,2} f$ on B.

Let X^i be uniform motion to the left, and let $\varphi^1 = \varphi^2$ be a bounded, odd, continuously differentiable function having a strictly negative derivative.[8] Then $B = \{x: x^1 + x^2 > 0\} = \{s, z: s > 0\}$. In this case, the class of proper functions contains $C_c^2(\mathbb{R}^2)$. For $f \in C_c^2(\mathbb{R}^2)$, formula (1.16) becomes

$$f(x) = \frac{1}{2}f(x^1, -x^1) + \frac{1}{2}f(-x^2, x^2) + \frac{1}{2}\int_{\Gamma_x} \frac{df}{dn}\, d\mu$$
$$+ \int_{\substack{y \in B \\ y \leqslant x}} \frac{\partial^2 f}{\partial y^1\, \partial y^2}(y)\, dy, \tag{1.18}$$

where Γ_x is the portion of ∂B lying between $(x^1, -x^1)$ and $(-x^2, x^2)$, $\partial f/dn$ is the derivative of f in the direction normal to ∂B and pointing into B, and μ is arc length along ∂B. The validity of (1.18) is easy to verify directly.

1.9. The application of stochastic waves to the investigation of X-harmonic functions is based on the following result proved in Section 4.2.

THEOREM 1.2. *For $\tilde{X}$-proper functions F, the following are equivalent*:

(a) *F is X-harmonic in B.*

(b) *F is $\tilde{X}$-harmonic in B.*

(c) *$\tilde{A}^{1,2}F = 0$ in B.*

[8]For example, $\varphi^i(x^i) = -\tan^{-1} x^i$.

If F is an $\tilde{X}$-proper function which is X-harmonic in B and satisfies

$$F = f, \quad \tilde{A}^1 F = \tilde{A}^1 f, \quad \tilde{A}^2 F = \tilde{A}^2 f, \quad x \in B^c, \tag{1.19}$$

then, by Theorems 1.1 and 1.2,

$$F(x) = P_x\left\{\frac{1}{2} f\left(\tilde{x}_{\Phi_-(x),0}\right) + \frac{1}{2} f\left(\tilde{x}_{0,\Phi_-(x)}\right) - \int_{L_x} \tilde{A}_N f(\tilde{x}_t)\, dt^1\right\}. \tag{1.20}$$

The converse is also true.

THEOREM 1.3. *For every $\tilde{X}$-proper function f, the function defined by* (1.20) *is $\tilde{X}$-proper, X-harmonic in B, and satisfies* (1.19).

This is proved in Section 4.2.

It follows from Theorems 1.1, 1.2, and 1.3 that, for every $\tilde{X}$-proper function f, there exists one and only one $\tilde{X}$-proper function F which is X-harmonic in B and satisfies (1.19).

Let's return to the case where X^i is uniform motion to the left and $B = \{x\colon x^1 + x^2 > 0\}$ (discussed in Section 1.8). If $f \in C_c^2(\mathbb{R}^2)$, (1.20) becomes

$$F(x) = \frac{1}{2}\left\{f(x^1, -x^1) + f(-x^2, x^2) + \int_{\Gamma_x} \frac{df}{dn}\, d\mu\right\}.$$

In terms of s and z,

$$F(s, z) = \frac{1}{2}\left\{ f(0, z - s) + f(0, z + s) + \int_{z-s}^{z+s} \frac{df}{ds}(0, y)\, dy\right\}.$$

This is the well known solution of the initial value problem for the one dimensional wave equation (see, e.g., [6] p. 39).

1.10. For an admissible set B, let

$$\partial B = \{y\colon \Phi(y) = 0\}.$$

From (1.5) and (1.20) it is easy to see that the value of the function F at the point x depends on the values of f and $\tilde{A}_N f$ on

$$D_x = \{y \in \partial B\colon \varphi^i(y^i) \geqslant \varphi^i(x^i), i = 1, 2\}.$$

By analogy with the wave equation, we call D_x the *domain of dependence* for the point x. For a point $y \in \partial B$, its *domain of influence* is

$$I_y = \{x \in B\colon \varphi^i(y^i) \geqslant \varphi^i(x^i), i = 1, 2\}.$$

1.11. In Section 4.3 we show that if an $\tilde{X}$-proper function f has the form

$$f(x) = P_x \int_{\mathrm{T}} \psi(x_t)\, dt, \tag{1.21}$$

then the function F given by (1.20) can be written as

$$F(x) = P_x \int_{\mathfrak{T}} \psi(x_t)\, dt, \tag{1.22}$$

where

$$\mathfrak{T} = \{t\colon x_s \notin B \text{ for some } 0 < s < t\}. \tag{1.23}$$

Formula (1.22) coincides with the formula proved in [3] for the case of time reversible processes.

1.12. In Section 2, we introduce some notations and definitions. We also give some examples of strong Markov processes and associated stochastic waves. Section 3 is devoted to proving some integral representations for X-proper functions. In the first part of Section 4 we use stochastic waves to study X-harmonic functions associated with a single strong Markov process X. The second part of Section 4 contains the proofs of Theorems 1.2 and 1.3.

2. Notations and Examples

2.1. Let $X = (\zeta, x_t, \mathfrak{F}_t, P_x, \theta_t)$ be a strong Markov process with state space $(E, \mathfrak{B})$ and sample space $(\Omega, \mathfrak{F})$ (the definition can be found in [2]). The notation $f \in \mathfrak{B}$ means that f is a bounded $\mathfrak{B}$-measurable function. This class of functions is a Banach space with norm

$$\|f\| = \sup|f(x)|.$$

The *semigroup* T_t, *Green's operator* G, and *weak infinitesimal operator* A are defined by the formulas

$$T_t f(x) = P_x f(x_t), \qquad t \geqslant 0, \tag{2.1}$$

$$Gf(x) = P_x \int_0^\infty f(x_t)\, dt, \tag{2.2}$$

$$Af(x) = w - \lim_{t \searrow 0} \frac{T_t f(x) - f(x)}{t}. \tag{2.3}$$

The semigroup is defined for all $f \in \mathfrak{B}$ and the Green's operator is defined for all $\mathfrak{B}$-measurable f for which $G|f|(x) < \infty$ for all x. The *domain* D of A

consists of those functions in $\mathfrak{B}$ for which the weak limit exists and is in

$$\mathfrak{B}_0 = \left\{ f \in \mathfrak{B} \colon w - \lim_{t \searrow 0} T_t f = f \right\}.$$

Functions in $\mathfrak{B}_0$ are called *T_t-continuous.* When we wish to emphasize the process X we will write $D(X)$ for the domain of A.

A very basic and useful result is the "fundamental theorem of calculus" in the Banach space $\mathfrak{B}$: for every $f \in D$,

$$T_t f - f = \int_0^t T_s A f \, ds. \tag{2.4}$$

By Fubini's theorem, this is equivalent to

$$f(x) = P_x \left\{ f(x_t) - \int_0^t A f(x_s) \, ds \right\}, \quad x \in E. \tag{2.5}$$

The stochastic wave introduced in Section 1.6 is the strong Markov process $\tilde{X} = (\tilde{\zeta}, \tilde{x}_t, \tilde{\mathfrak{F}}_t, P_x, \tilde{\theta}_t)$, where

$$\tilde{x}_t = x_{\sigma_t}, \quad \tilde{\mathfrak{F}}_t = \mathfrak{F}_{\sigma_t}, \quad \tilde{\theta}_t = \theta_{\sigma_t}, \quad \tilde{\zeta} = \inf\{t > 0 \colon \sigma_t \geqslant \zeta\}.$$

The state space and sample space for $\tilde{X}$ are the same as for X. Stochastic waves are studied in [4].

2.2. Examples

Uniform motion to the right with unit velocity: $U = \{u_t\}_{t \geqslant 0}$. The domain $D(U)$ consists of all bounded continuous functions f having a bounded right continuous right hand derivative f^+. For $f \in D(U)$, $Af = f^+$.

Brownian motion in $\mathbb{R}^d$: $W = \{w_t\}_{t \geqslant 0}$. The domain $D(W)$ contains all bounded twice continuously differentiable functions whose second derivatives are bounded. For $d = 1$, this is the exact description of $D(W)$. For functions $f \in D(W)$ which are twice continuously differentiable, $Af = (1/2)\Delta f$.

Any stochastic wave: $\tilde{X} = \{\tilde{x}_t\}_{t \geqslant 0}$. It follows from (1.10) that $\varphi(\tilde{x}_t)$ is uniform motion to the right with unit velocity. This implies that, for every $h \in D(U)$, the function $f = h \circ \varphi$ belongs to $\tilde{D}$ and $\tilde{A} f = h^+ \circ \varphi$.

The stochastic wave corresponding to Brownian motion in $\mathbb{R}^d$ and $\varphi(x) = |x|$. Computations show that every twice continuously differentiable function with compact support is contained in $\tilde{D}$ and every function in $\tilde{D}$ is continuous. In addition, if $f \in \tilde{D}$ is twice continuously differentiable, then

$$\begin{aligned} \tilde{A} f(x) &= \frac{df}{dn}(x) - \int_{|y| = |x|} \frac{2}{\omega_d} \frac{f(y) - f(x)}{|y - x|^d} \, dS_y, & x \neq 0, \\ &= 0, & x = 0, \end{aligned}$$

where df/dn is the derivative of f in the direction of the outward directed normal to $\{y: |y| = |x|\}$, ω_d is the surface area of the unit ball in $\mathbb{R}^d$, S_y is surface area, and the integral is a principle value integral. For $d = 1$, an exact description can be given. Indeed, $\tilde{D}$ consists of all bounded continuous functions which have a bounded right continuous right hand derivative for $x > 0$, a bounded left continuous left hand derivative for $x < 0$, and satisfy

$$\lim_{x \searrow 0} \frac{1}{2}(f^+(x) - f^-(-x)) = \lim_{x \searrow 0} \frac{f(x) - 2f(0) + f(-x)}{2x}.$$

For $f \in \tilde{D}$,

$$\begin{aligned} \tilde{A}f(x) &= \frac{df}{dn}(x) - \frac{f(x) - f(-x)}{2|x|}, & x &\neq 0, \\ &= \lim_{x \searrow 0} \frac{1}{2}(f^+(x) - f^-(-x)), & x &= 0, \end{aligned}$$

where

$$\begin{aligned} \frac{df}{dn}(x) &= f^+(x), & x &> 0, \\ &= f^-(x), & x &< 0. \end{aligned}$$

2.3. For $i = 1, 2$, let $X^i = (\zeta^i, x^i_{t^i}, \mathcal{F}^i_{t^i}, P^i_{x^i}, \theta^i_{t^i})$ be a strong Markov process with sample space $(\Omega^i, \mathcal{F}^i)$ and state space $(E^i, \mathcal{B}^i)$. Put

$$\begin{aligned} &(\Omega, \mathcal{F}) = (\Omega^1, \mathcal{F}^1) \times (\Omega^2, \mathcal{F}^2), \quad (E, \mathcal{B}) = (E^1, \mathcal{B}^1) \times (E^2, \mathcal{B}^2), \\ &\quad \mathbf{T} = [0, \infty)^2 \quad \omega = (\omega^1, \omega^2) \quad x = (x^1, x^2), \quad t = (t^1, t^2), \\ &\quad \zeta = (\zeta^1, \zeta^2), \quad x_t = (x^1_{t^1}, x^2_{t^2}), \quad \mathcal{F}_t = \mathcal{F}^1_{t^1} \times \mathcal{F}^2_{t^2}, \quad P_x = P^1_{x^1} \times P^2_{x^2}, \\ &\quad \theta_t = \theta^1_{t^1}\theta^2_{t^2}. \end{aligned}$$

The collection $X = (\zeta, x_t, \mathcal{F}_t, P_x, \theta_t)$ is called a *pair of independent strong Markov processes* with sample space $(\Omega, \mathcal{F})$ and state space $(E, \mathcal{B})$. We will also write $X = (X^1, X^2)$.

The two-parameter semigroup T_t and the various infinitesimal operators were defined in Section 1.7. The Green's operator for $X = (X^1, X^2)$ is defined by

$$Gf(x) = P_x \int_{\mathbf{T}} f(x_t)\, dt$$

for functions f for which $G|f|(x) < \infty$ for all x. It is easy to see that $G = G^1 G^2$, where G^i is the Green's operator for X^i acting on f as a function of x^i.

3. INTEGRAL REPRESENTATIONS OF X-PROPER FUNCTIONS

3.1. First let's prove the remark in Section 1.7 that, for X-proper functions, the operators A^1 and A^2 commute. Let f be an X-proper function, and put $g^{ij} = A^i A^j f$. By (2.4) and Fubini's theorem,

$$\begin{aligned}(T^1_{t^1} - \mathbf{1})(T^2_{t^2} - \mathbf{1})f &= \int_{(0,\,t\,]} T_s g^{12}\, ds \\ &= \int_{(0,\,t\,]} T_s g^{21}\, ds, \end{aligned}\tag{3.1}$$

where $(0, t] = (0, t^1] \times (0, t^2]$. Hence

$$\int_{(0,\,t\,]} T_s\big(g^{12} - g^{21}\big)\, ds = 0.$$

Dividing by $t^1 t^2$, letting $t \searrow 0$, and using the fact that $g^{12} - g^{21} \in \mathfrak{B}_0$, we see that $g^{12} = g^{21}$.

It follows from (3.1) that

$$f(x) = P_x\left\{ f(x_{t^1,0}) + f(x_{0,t^2}) - f(x_t) + \int_{(0,\,t\,]} A^{1,2} f(x_s)\, ds \right\}. \tag{3.2}$$

THEOREM 3.1. *If $f \in D^{1,2}$, then for every $x \in E$, $S \geqslant 0$,*

$$f(x) = P_x\left\{ f(x_{S,0}) - \int_{\substack{0 \\ t^1+t^2=S}}^{S} A^1 f(x_t)\, dt^1 + \int_{R(S)} A^{1,2} f(x_t)\, dt \right\} \tag{3.3}$$

$$= P_x\left\{ f(x_{0,S}) - \int_{\substack{0 \\ t^1+t^2=S}}^{S^1} A^2 f(x_t)\, dt^2 + \int_{R(S)} A^{1,2} f(x_t)\, dt \right\}, \tag{3.4}$$

where $R(S) = \{t = (t^1, t^2) \in \mathbf{T}\colon t^1 + t^2 \leqslant S\}$. Averaging (3.3) *and* (3.4) *gives a symmetric formula*:

$$\begin{aligned} f(x) = P_x\Bigg\{ &\frac{1}{2} f(x_{S,0}) + \frac{1}{2} f(x_{0,S}) - \int_{\substack{0 \\ t^1+t^2=S}}^{S} A_N f(x_t)\, dt^1 \\ &+ \int_{R(S)} A^{1,2} f(x_t)\, dt \Bigg\}, \end{aligned}\tag{3.5}$$

where

$$A_N f = \tfrac{1}{2}(A^1 f + A^2 f).$$

Proof. It suffices to prove (3.3) for $S > 0$. Fix $x \in E$. Put $h(t) = T_t f(x)$, $h^1(t) = T_t A^1 f(x)$ and $h^{1,2}(t) = T_t A^{1,2} f(x)$. For $s < t \in \mathbf{T}$, let

$$\begin{aligned} h(s,t] &= h(t) - h(s^1, t^2) - h(t^1, s^2) + h(s) \\ &= \left(T^1_{t^1 - s^1} - 1\right)\left(T^2_{t^2 - s^2} - 1\right) h(s). \end{aligned} \tag{3.6}$$

By (3.1),

$$h(s,t] = \int_{(s,t]} h^{1,2}(r)\, dr. \tag{3.7}$$

Let $\mathbf{P}$ be a partition: $0 = t_0^1 < t_1^1 < \cdots < t_N^1 = S$. Let $\|\mathbf{P}\| = \max|t^1_{n+1} - t^1_n|$. Put $t_n^2 = S - t_n^1$, $t_n = (t_n^1, t_n^2)$ and $t_n^* = (t^1_{n+1}, t^2_n)$. Let $Q(\mathbf{P}) = \cup_n (0, t_n^*]$. Clearly $\cap_{\mathbf{P}} Q(\mathbf{P}) = \mathrm{R(S)}$. From (3.7) we have

$$\begin{aligned} \int_{Q(\mathbf{P})} h^{1,2}(r)\, dr &= \sum_{n=0}^{N-1} \{h(t_n^*) - h(t_n)\} - h(t_N) + h(0) \\ &= \sum_{n=0}^{N-1} \int_{\Delta_n} h^1(s^1, t_n^2)\, ds^1 - h(t_N) + h(0) \\ &= \int_0^S h^1\left(s^1, u(s^1, \mathbf{P})\right) ds^1 - h(t_N) + h(0), \end{aligned} \tag{3.8}$$

where Δ_n is the interval between t_n^1 and t_{n+1}^1 and $u(s^1, \mathbf{P}) = t_n^2$ for $t_n^1 < s^1 \leqslant t^1_{n+1}$. Let $\mathbf{P}_m$ be a sequence of partitions for which $\|\mathbf{P}_m\| \to 0$ as $m \to \infty$. Obviously, $\cap_m Q(\mathbf{P}_m) = R(S)$ and $u(s^1, \mathbf{P}_m) \to S - s^1$. By the dominated convergence theorem,

$$\int_{Q(\mathbf{P}_m)} h^{1,2}(r)\, dr \to \int_{R(S)} h^{1,2}(r)\, dr. \tag{3.9}$$

Since $A^1 f \in \mathcal{B}_0$,

$$\int_0^S h^1\left(s^1, u(s^1, \mathbf{P}_m)\right) ds^1 \to \int_{\substack{0 \\ s^1 s^2 = S}}^{S} h^1(s^1, s^2)\, ds^1. \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), we get (3.3).

4. X-Harmonic Functions

4.1 In this section we use the stochastic wave $\tilde{X}$, corresponding to a strong Markov process X and a function φ, to investigate functions which are X-harmonic in the set

$$B = \{x: \varphi(x) < 0\}.$$

Indeed, we will show that, for every $f \in \tilde{D}$, there exists one and only one function $F \in \tilde{D}$ which is X-harmonic in B and satisfies $F = f$ on B^c. We call F the *solution of the Dirichlet problem for X*.

The methods presented here serve as a prelude to Section 4.2, where we prove the analogous results for a pair of processes.

From (1.10) we see that

$$\tilde{\tau}(B) = \varphi_-(x_0), \tag{4.1}$$

where $\tilde{\tau}(B)$ is the first exit time of $\tilde{x}_t$ from B. Also, $\tilde{x}_t \in B$ if and only if $t < \tilde{\tau}(B)$.

The following lemma shows that X and $\tilde{X}$ have the same exit distributions[10] from the sets $\{x: \varphi(x) < u\}$.

LEMMA 4.1. $\sigma_{\tilde{\tau}(B)} = \tau(B)$ a.s. P_x *for all* $x \in E$. *Hence, for every* $f \in \mathfrak{B}$,

$$P_x f(x_{\tau(B)}) = P_x f(\tilde{x}_{\tilde{\tau}(B)}).$$

Proof. For $x \in B^c$, both stopping times are zero, so consider $x \in B$. Put $\tau = \tau(B)$ and $\tilde{\tau} = \tilde{\tau}(B)$. Since $\sigma_{\tilde{\tau}} = \inf\{t > 0: \varphi(x_t) > 0\}$, $\tau \leqslant \sigma_{\tilde{\tau}}$, and $\sigma_{\tilde{\tau}} = \theta_\tau \sigma_{\tilde{\tau}} + \tau$. By the strong Markov property, $P_x \theta_\tau \sigma_{\tilde{\tau}} = P_x P_{x_\tau} \sigma_{\tilde{\tau}}$. By 1.4.A, $\varphi(x_\tau) = 0$ a.s. P_x. Hence $\sigma_{\tilde{\tau}} = \sigma_0$ a.s. P_{x_τ}, and so $P_x \theta_\tau \sigma_{\tilde{\tau}} = 0$.

THEOREM 4.1. *For functions* $F \in \tilde{D}$, *the following are equivalent*:

(a) *F is X-harmonic in B.*

(b) *F is $\tilde{X}$-harmonic in B.*

(c) $\tilde{A}F = 0$ *in* B.

Proof. We prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). Suppose F is X-harmonic in B. Let $\tilde{\tau}$ be a stopping time for $\tilde{X}$ such that $\tilde{\tau} \leqslant \tilde{\tau}(B)$. Put $\tau = \sigma_{\tilde{\tau}}$. Then $\tau \leqslant \sigma_{\tilde{\tau}(B)}$. By Lemma 4.1, $\tau \leqslant \tau(B)$. Hence, $P_x F(\tilde{x}_{\tilde{\tau}}) = P_x F(x_\tau) = F(x)$.

Now suppose that F is $\tilde{X}$-harmonic in B. Fix $x \in B$. For $t < \varphi_-(x)$,

$$\begin{aligned} 0 &= P_x F(\tilde{x}_t) - F(x) \\ &= \int_0^t \tilde{T}_s g(x)\, ds, \end{aligned}$$

[10] It is well known (see [1, p. 234]) that, within a large class of processes, two processes have the same exit distribution from all sets in $\mathfrak{B}$ if and only if one is a random time change of the other, and this random time change is the inverse of a continuous strictly increasing additive functional. This is consistent with the fact that σ_t is the inverse of $\gamma_t = \sup_{0 \leqslant s \leqslant t}(\varphi(x_s) - \varphi(x_0))$, which is not an additive functional.

where $g = \tilde{A}F$. Dividing by t and letting $t \searrow 0$ we get $g(x) = 0$ (since $g \in \tilde{\mathfrak{B}}_0$). Hence $\tilde{A}F = 0$ in B.

Finally, suppose $\tilde{A}F = 0$ in B. By (2.4),

$$P_x F\left(\tilde{x}_{\varphi_-(x)}\right) - F(x) = 0,$$

and so, by Lemma 4.1,

$$F(x) = P_x F(x_{\tau(B)}). \tag{4.2}$$

From (4.2) it is easy to see that F is X-harmonic in B. This completes the proof.

If F is a function in $\tilde{D}$ which is X-harmonic in B and satisfies $F = f$ on B^c, then by Theorem 4.1, Lemma 4.1 and (2.4),

$$F(x) = P_x f(x_{\tau(B)}). \tag{4.3}$$

The converse is also true:

THEOREM 4.2. *If $f \in \tilde{D}$, the function defined by* (4.3) *is in $\tilde{D}$, is X-harmonic in B, and satisfies $F = f$ on B^c. Furthermore,*

$$\begin{aligned} \tilde{A}F &= \tilde{A}f, \qquad x \in B^c,\\ &= 0, \qquad x \in B. \end{aligned}$$

It is not true that $F \in D$, even if we assume that $f \in D$.

Proof. Put $\tilde{\tau} = \tilde{\tau}(B)$. From (4.1) it is easy to see that $\tilde{\theta}_t\tilde{\tau} + t = \tilde{\tau} \vee t$. From this we get

$$\tilde{T}_t F(x) - F(x) = P_x\{f(\tilde{x}_{\tilde{\tau}\vee t}) - f(\tilde{x}_{\tilde{\tau}})\}. \tag{4.4}$$

We note that $\tilde{\tau} \vee t - \tilde{\tau} = (t - \tilde{\tau})_+$. Hence, by the strong Markov property,

$$\tilde{T}_t F(x) - F(x) = P_x P_{\tilde{x}_{\tilde{\tau}}}\left\{f\left(\tilde{x}_{(t-\tilde{\tau})_+}\right) - f(\tilde{x}_0)\right\}.$$

This implies that

$$\begin{aligned} \|\tilde{T}_t F - F\| &\leqslant \sup_{0\leqslant s\leqslant t}\ \sup_y |P_y f(\tilde{x}_s) - f(y)|\\ &= \sup_{0\leqslant s\leqslant t} \|\tilde{T}_s f - f\|, \end{aligned}$$

and so $\|\tilde{T}_t F - F\|/t$ is bounded.

Fix $x \in B$. By (4.4) and (4.1), $\tilde{T}_t F(x) - F(x) = 0$ for $t < \varphi_-(x)$, and so $(\tilde{T}_t F(x) - F(x))/t \to 0$ as $t \searrow 0$. Now take $x \in B^c$. Then $\tilde{\tau} = 0$, and by

(4.4) we see that $\tilde{T}_t F(x) - F(x) = \tilde{T}_t f(x) - f(x)$, and so $(\tilde{T}_t F(x) - F(x))/t \to \tilde{A} f(x)$ as $t \searrow 0$. Put

$$\begin{aligned} g &= \tilde{A} f, & x &\in B^c, \\ &= 0, & x &\in B. \end{aligned}$$

The proof will be complete once we show that $w - \lim_{t \searrow 0} \tilde{T}_t\, g = g$. Since $\|\tilde{T}_t g\| \leqslant \|g\|$ for all t, we need only to show pointwise convergence. Fix $x \in B$. For $t \leqslant \varphi_-(x)$, $\tilde{T}_t g(x) = 0$. Now take $x \in B^c$. Then $\tilde{T}_t g(x) = P_x g(\tilde{x}_t) = P_x \tilde{A} f(\tilde{x}_t) = \tilde{T}_t(\tilde{A} f)(x) \to \tilde{A} f(x) = g(x)$ as $t \searrow 0$.

4.2. Now let $X = (X^1, X^2)$ be a pair of independent strong Markov processes, φ^1 and φ^2 be bounded functions satisfying 1.4.A, B and let $\tilde{X}^i$ be the stochastic wave corresponding to X^i and φ^i. In this section we will prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. We prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). Suppose F is X-harmonic in B. Let $\tilde{\tau}$ be a stopping vector for $\tilde{X}$ such that $\tilde{x}_t \in B$ for $0 \neq t \leqslant \tilde{\tau}$. Put $\tau = \sigma_{\tilde{\tau}}$. Suppose $\tilde{x}_0 \in B$. Then $\tilde{\tau}^1 + \tilde{\tau}^2 < -\Phi(\tilde{x}_0)$ and so, for $0 \neq t \leqslant \tau$, $\Phi(x_t) \leqslant \Phi(x_\tau) = \Phi(\tilde{x}_{\tilde{\tau}}) = \Phi(\tilde{x}_0) + \tilde{\tau}^1 + \tilde{\tau}^2 < 0$. Hence $x_t \in B$ for $0 \neq t \leqslant \tau$ on $\{\omega : \tilde{x}_0 \in B\}$. For $\tilde{x}_0 \in B^c$, $\tilde{\tau}^1 + \tilde{\tau}^2 = 0$. Hence $\tau = 0$ and $\{t : 0 \neq t \leqslant \tau\}$ is empty. Therefore $x_t \in B$ for $0 \neq t \leqslant \tau$. The $\tilde{X}$-harmonicity now follows from X-harmonicity.

Now suppose that F is $\tilde{X}$-harmonic in B. Fix $x \in B$. For t small enough so that $t^1 + t^2 < \Phi_-(x)$,

$$\begin{aligned} 0 &= P_x\{F(\tilde{x}_t) - F(\tilde{x}_{t^1,0}) - F(\tilde{x}_{0,t^2}) + F(\tilde{x}_0)\} \\ &= \int_{(0,t]} \tilde{T}_s g(x)\, ds, \end{aligned}$$

where $g = \tilde{A}^{1,2} F$. Divide by $t^1 t^2$, and let $t \searrow 0$ to get $g(x) = 0$ (since $g \in \tilde{\mathfrak{B}}_0$). Hence $\tilde{A}^{1,2} F = 0$ in B.

Finally, suppose that $\tilde{A}^{1,2} F = 0$ in B. By Theorem 3.1,

$$F(x) = P_x\left\{F\left(\tilde{x}_{0,\Phi_-(x)}\right) - \int_{L_x} \tilde{A}_N F(\tilde{x}_t)\, dt^1\right\}.$$

The fact that F is X-harmonic in B now follows from the next Lemma.

Lemma 4.2. *The functions* $F(x) = P_x f(\tilde{x}_{0,\Phi_-(x)})$ *and* $H(x) = P_x \int_{L_x} g(\tilde{x}_t)\, dt^1$ *are X-harmonic in B.*

Proof. First consider F. Put $\eta = \sigma^2_{\Phi_-(x_0)}$. For any stopping vector τ,

$$P_x F(x_\tau) = P_x P_{x_\tau} f\left(\tilde{x}_{0,\Phi_-(x_0)}\right) = P_x \theta_\tau f(x_{0,\eta}) = P_x f(x_{\tau^1,\bar{\eta}}), \quad (4.5)$$

where $\bar{\eta} = \tau^2 + \theta_\tau \eta$. By the definition of $\sigma_{t^2}^2$,

$$\eta = \inf\{u > 0: \Phi(x_{0,u}) > 0\},$$

and so,

$$\theta_\tau \eta = \inf\{u > \tau^2: \Phi(x_{\tau^1, u}) > 0\} - \tau^2.$$

If τ is a stopping vector for which $x_t \in B$ for $0 \neq t \leqslant \tau$, then $\Phi(x_{\tau^1, u}) < 0$ for $0 < u \leqslant \tau^2$. Hence,

$$\theta_\tau \eta = \theta_{\tau^1}^1 \eta - \tau^2.$$

Consequently,

$$\bar{\eta} = \theta_{\tau^1}^1 \eta,$$

and (4.5) becomes

$$\begin{aligned} P_x F(x_\tau) &= P_x \theta_{\tau^1}^1 f(x_{0,\eta}) = P_x P_{x_{\tau,0}} f(x_{0,\eta}) \\ &= P_x F(x_{\tau^1, 0}). \end{aligned} \tag{4.6}$$

Replacing τ by $(0, \tau^2)$ we get

$$P_x F(x_{0,\tau^2}) = P_x F(x_0). \tag{4.7}$$

Combining (4.6) and (4.7) we see that F is X-harmonic in B.

Now consider H. Put $v_{t^i}^i = \varphi^i(x_{t^i}^i) - \varphi^i(x_0^i)$. For any stopping vector τ,

$$P_x H(x_\tau) = P_x \int_{\substack{-\infty \\ t^1 + t^2 = \Phi_-(x)}}^{\infty} V_1 V_2 g(x_{\bar{\sigma}_t})\, dt^1, \tag{4.8}$$

where V_i is the indicator function of the set $\{t, \omega: t^i \geqslant v_{\tau^i}^i\}$, $\bar{\sigma}_t = (\bar{\sigma}_{t^1}^1, \bar{\sigma}_{t^2}^2)$, and

$$\bar{\sigma}_{t^i}^i = \inf\{u > \tau^i: v_u^i > t^i\}. \tag{4.9}$$

Put $\nu_t^1 = (\bar{\sigma}_{t^1}^1, \sigma_{t^2}^2)$ and $\nu_t^2 = (\sigma_{t^1}^1, \bar{\sigma}_{t^2}^2)$. From (4.9) and the definition of $\sigma_{t^i}^i$,

$$\{\sigma_{t^i}^i > \tau^i\} \subset \{\bar{\sigma}_{t^i}^i = \sigma_{t^i}^i\} \cap \{v_{\tau^i}^i \leqslant t^i\}. \tag{4.10}$$

If $x_t \in B$ for $0 \neq t \leqslant \tau$, then $\{\sigma_t \leqslant \tau\} = \varnothing$ for $t^1 + t^2 = \Phi_-(x_0)$. Let W_i be the indicator function of the set $\{t, \omega: \tau^i \geqslant \sigma_{t^i}^i\}$. Taking into account (4.9) and (4.10), we can write (4.8) as

$$\begin{aligned} P_x H(x_\tau) = P_x \int_{\substack{-\infty \\ t^1 + t^2 = \Phi_-(x)}}^{\infty} & \{(1 - W_1)(1 - W_2) g(x_{\sigma_t}) + W_1(1 - W_2) V_1 g(x_{\nu_t^1}) \\ & + (1 - W_1) W_2 V_2 g(x_{\nu_t^2})\}\, dt^1. \end{aligned}$$

Replacing τ by $(\tau^1, 0)$ and $(0, \tau^2)$, we get

$$P_x\{H(x_\tau) - H(x_{\tau^1,0}) - H(x_{0,\tau^2}) + H(x_0)\}$$
$$= P_x \int_{\substack{-\infty \\ t^1+t^2=\Phi_-(x)}}^{\infty} W_1 W_2\{g(x_{\sigma_t}) - V_1 g(x_{\nu_t^1}) - V_2 g(x_{\nu_t^2})\}\, dt^1.$$

But $W_1 W_2 = 0$, and so H is X-harmonic in B.

Before embarking on the proof of Theorem 1.3, we need a couple of lemmas.

LEMMA 4.3. *For every $h \in \tilde{\mathfrak{B}}_0$, the function*

$$H(x) = P_x \int_{R(x)} h(\tilde{x}_s)\, ds \tag{4.11}$$

is in $\tilde{D}^1$ and $\tilde{D}^2$. Furthermore

$$\tilde{A}^1 H(x) = -P_x \int_0^{\Phi_-(x)} h(\tilde{x}_{0,u})\, du,\ \tilde{A}^2 H(x) = -P_x \int_0^{\Phi_-(x)} h(\tilde{x}_{u,0})\, du.$$

Proof. It is easy to check that

$$\tilde{T}_t H(x) = \int_{R(x) \cap \{s \geqslant t\}} \tilde{T}_s h(x)\, ds. \tag{4.12}$$

H is bounded since

$$\|H\| \leqslant \tfrac{1}{2}\|h\|\,\|\Phi_-\|^2.$$

Hence $\tilde{T}_t H$ is bounded. By the dominated convergence theorem and (4.12), $\tilde{T}_t H$ tends to H pointwise. Hence $H \in \tilde{\mathfrak{B}}_0$.

Now,

$$\tilde{T}_t^1 H(x) - H(x) = -\int_{R(x) \cap \{s^1 < t^1\}} \tilde{T}_s h(x)\, ds.$$

From this we see that

$$\|\tilde{T}_t^1 H - H\| \leqslant \|h\|\,\|\Phi_-\| t^1,$$

and so $\|\tilde{T}_t^1 H - H\|/t^1$ is bounded. For $x \in B^c$, $\tilde{T}_t^1 H = 0$ for all $t^1 \geqslant 0$ and so $(\tilde{T}_t^1 H(x) - H(x))/t^1$ converges to zero as $t^1 \searrow 0$. Fix $x \in B$. If $t^1 < \Phi_-(x)$,

$$\tilde{T}_t^1 H(x) - H(x) = \int_{A_{t,x}} \tilde{T}_s h(x)\, ds + \int_{B_{t,x}} \tilde{T}_s h(x)\, ds, \tag{4.13}$$

where $A_{t,x} = \{s: s^1 + s^2 \geqslant \Phi_-(x), s^1 < t^1, s^2 < \Phi_-(x)\}$ and $B_{t,x} = (0, t^1] \times (0, \Phi_-(x)]$. The measure of $A_{t,x}$ is $1/2\,(t^1)^2$, and so

$$\frac{1}{t^1}\left|\int_{A_{t,x}} \tilde{T}_s h(x)\, ds\right| \leqslant \|h\|\frac{t^1}{2}.$$

Hence the first term in (4.13) tends to zero as $t^1 \searrow 0$. By the dominated convergence theorem and the fact that $h \in \tilde{\mathfrak{B}}_0$,

$$\lim_{t^1\searrow 0}\frac{1}{t^1}\left(\tilde{T}^1_t H(x) - H(x)\right) = -\int_0^{\Phi_-(x)}\tilde{T}_{0,u}h(x)\, du$$
$$= -P_x\int_0^{\Phi_-(x)} h(\tilde{x}_{0,u})\, du.$$

Put $g(x) = -P_x\int_0^{\Phi_-(x)} h(\tilde{x}_{0,u})\, du$. For $x \in B^c$, $\tilde{T}_t g(x) = 0$ for all $t \geqslant 0$. Fix $x \in B$. If $t^1 + t^2 < \Phi_-(x)$, then

$$\tilde{T}_t g(x) = -\int_0^{\Phi_-(x)}\tilde{T}_{0,u}\tilde{T}_t h(x)\, du + \int_{\Phi_-(x)-t^1-t^2}^{\Phi_-(x)}\tilde{T}_{0,u}\tilde{T}_t h(x)\, du.$$

The second term is bounded by $\|h\|(t^1 + t^2)$ which tends to zero as t does. By the dominated convergence theorem and the fact that $h \in \tilde{\mathfrak{B}}_0$, the first term tends to $g(x)$. Hence $g \in \tilde{\mathfrak{B}}_0$ and $H \in \tilde{D}^1$. The proof that $H \in \tilde{D}^2$ is the same.

LEMMA 4.4. *Let*

$$H_1(x) = -P_x\int_0^{\Phi_-(x)} h(\tilde{x}_{u,0})\, du,\; H_2(x) = -P_x\int_0^{\Phi_-(x)} h(\tilde{x}_{0,u})\, du.$$

If $h \in \tilde{\mathfrak{B}}_0$, then $H_i \in \mathfrak{D}^i$ and

$$\tilde{A}^i H_i = 0 \qquad x \in B^c,$$
$$= h, \qquad x \in B.$$

Proof. It suffices to consider $H(x) = H_2(x)$. It is easy to check that

$$\tilde{T}^2_{t^2} H(x) - H(x) = 1_{t^2 < \Phi_-(x)} P_x\int_0^{t^2} h(\tilde{x}_{0,u})\, du$$
$$- 1_{t^2 \geqslant \Phi_-(x)} P_x\int_{t^2}^{\Phi_-(x)} h(\tilde{x}_{0,u})\, du.$$

Each of the terms on the right are bounded by $\|h\|t^2$, and so $\|\tilde{T}^2_{t^2}H - H\|/t^2$ is bounded. For $x \in B^c$, $(\tilde{T}^2_{t^2}H(x) - H(x))/t^2 = 0$ for all $t^2 > 0$. Fix $x \in B$. If $t^2 < \Phi_-(x)$,

$$\frac{\tilde{T}^2_{t^2}H(x) - H(x)}{t^2} = \frac{1}{t^2}P_x\int_0^{t^2} h(\tilde{x}_{0,u})\, du \to h(x)$$

as $t^2 \searrow 0$ (since $h \in \tilde{\mathfrak{B}}_0$).

Put

$$g = 0, \qquad x \in B^c,$$
$$= h, \qquad x \in B.$$

For $x \in B^c$, $\tilde{T}_t g = 0$ for all $t \geqslant 0$. Fix $x \in B$. If $t^1 + t^2 < \Phi_-(x)$, then $\tilde{T}_t g(x) = \tilde{T}_t h(x) \to h(x) = g(x)$ as $t \searrow 0$. Hence $g \in \tilde{\mathfrak{B}}_0$.

Theorem 1.3 is an abridged version of

THEOREM 4.3. *If f is an $\tilde{X}$-proper function, then the function F defined by (1.20) is also $\tilde{X}$-proper. For $x \in B^c$, $F(x) = f(x)$, $\tilde{A}^1F(x) = \tilde{A}^1f(x)$, $\tilde{A}^2F(x) = \tilde{A}^2f(x)$, and $\tilde{A}^{1,2}F(x) = \tilde{A}^{1,2}f(x)$. For $x \in B$,*

$$\tilde{A}^1F(x) = P_x\tilde{A}^2f\left(\tilde{x}_{0,\Phi_-(x)}\right), \tilde{A}^2F(x) = P_x\tilde{A}^1f\left(\tilde{x}_{\Phi_-(x),0}\right),$$
$$\tilde{A}^{1,2}F(x) = 0.$$

Proof. By Theorem 3.1 and the linearity of the class of $\tilde{X}$-proper functions, it suffices to show that, for every $h \in \tilde{\mathfrak{B}}_0$, the function H defined by (4.11) is $\tilde{X}$-proper and that

$$\tilde{A}^1H(x) = -P_x\int_0^{\Phi_-(x)} h(\tilde{x}_{0,u})\,du, \tilde{A}^2H(x) = -P_x\int_0^{\Phi_-(x)} h(\tilde{x}_{u,0})\,du,$$

$$\tilde{A}^{1,2}H(x) = 0, \qquad x \in B^c,$$
$$= h(x), \qquad x \in B.$$

These statements follow immediately from Lemmas 4.3 and 4.4.

4.3 In this section we prove the result described in Section 1.11.

Fix $x \in E$. By Theorem 3.1,

$$F(x) = P_x\left\{f\left(\tilde{x}_{\Phi_-(x),0}\right) - \int_{L_x}\tilde{A}^1f(\tilde{x}_t)\,dt^1\right\}. \tag{4.14}$$

Investigating the proof of Theorem 3.1, we see that the right hand side of (4.14) is a limit as a sequence of partitions $\mathbf{P}_m$ is refined in such a way that $\|\mathbf{P}_m\| \to 0$. Hence

$$F(x) = \lim_{m\to\infty}\left\{h(t_N) - \sum_{n=0}^{N-1}\left[h(t_n^*) - h(t_n)\right]\right\}$$

(t_n and t_n^* are defined in the proof of Theorem 3.1), where $h(t) = \tilde{T}_tf(x) = T_{\sigma_t}f(x) = P_x\int_{(\sigma_t,\infty)}\psi(x_s)\,ds$. It is easy to check that

$$h(t_N) - \sum_{n=0}^{N-1}\left[h(t_n^*) - h(t_n)\right] = P_x\int_{C_m}\psi(x_s)\,ds,$$

where $C_m = \{s: s > \sigma_t \text{ for some } t \in \mathbf{P}_m\}$. Applying the dominated convergence theorem, we get

$$F(x) = P_x \int_{\cup C_m} \psi(x_s)\, ds.$$

To complete the proof, note that $\cup C_m = \{s: x_u \notin B \text{ for some } 0 < u < s\}$.

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