Bewitching Switching Paradoxes

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1 Introduction

For hundreds of years paradoxes dealing with questions of probability have ranked among some of the most perplexing of all mathematical paradoxes. In this chapter, we will discuss in detail some interesting probabilistic paradoxes and at the end we will attempt to offer an explanation as to why probabilistic paradoxes tend such as these tend to be so perplexing.

2 The Let's Make a Deal Paradox

This paradox has been around for a long time, but recently it has generated a great deal of interest following its appearance in syndicated puzzler Marilyn Vos Savant's weekly brain teaser article that appears in many newspapers throughout the United States.

Here's how the paradox goes. In the game show *Let's Make a Deal*, the following scenario arises frequently. On the stage there are three large doors. The host, Monty Hall, tells the contestant that behind one of the doors is a nice prize but the other two doors have nothing of value behind them. The contestant is offered the chance to select one of the doors. Let's call this door door A. Before showing the contestant what is behind door A, Monty Hall, who knows which door actually conceals the nice prize, shows the contestant what is behind one of the two doors that the contestant didn't choose. Let's call this door door B and the door that he doesn't show you we will call door C. Monty Hall always picks for door B a door that does not have the prize. At this point, Monty asks the contestant whether he or she would like to switch to door C. Should the contestant switch? In other words, what is the probability that the prize is behind door A given that there is nothing behind door B?

When first presented with this question, most students (and professors) of probability say that it does not matter; either way, the probability of success is 1/2. After all, there are two choices that remain and knowing that door B is empty doesn't say anything about doors A and C. Right?

	1	2	3
H	2	3	2
T	3	3	2

Table 1: The columns represent the values of P. The two rows represent the coin that Monty flips to decide what to say in the case that $P = 1$. Each of the six sample points has probability 1/6. The numbers given in the table indicate which door Monty will show.

Actually, it's wrong. But the conviction that 1/2 is the correct answer can be very strong. In fact, on a recent visit to an Ivy League school one of us got into a rather intense discussion about this problem with three distinguished mathematicians (one of whom is even a probabilist). All three of them were convinced that 1/2 was the success probability whether they stick with their original door or switch.

All sorts of reasoning was applied to convice them of their error, but it was to no avail. Experimental evidence was offered first. They were told that a computer program had been written that simulates this game. The program was written so that the contestant always switches and was set to simulate 10 million plays of the game. Of the 10 million simulated contestants, only 3,332,420 of them lost. This gives an empirical success rate of 0.6667580. It was suggested that this number looks suspiciously close to 2/3. Of course, this piece of evidence did not alter their convictions. They prefered to believe that the program was flawed rather than their logic.

After experimental evidence failed, raw logic was employed. It was argued that the probability of success for door A was 1/3 before door B was revealed and since Monty Hall is always able to find some door to be door B, how can the opening of door B say anything about door A. Hence, its probability must still be 1/3. Since door C is the only remaining door, its success probability must then be 2/3. Amazingly, this still did not convince our three distinguished friends.

Finally, they were offered the best explanation. They were shown an actual definition of a probability space that models this game and were then shown the computations that lead to 1/3 and 2/3. Here is the model. First we need a random variable P that describes which door the prize is behind (i.e. $P = 1, 2$, or 3, each with probability 1/3). Without loss of generality, we may assume that the contestant always picks door 1 (so that door 1 is door A). Now, if the prize is behind door 3, Monty Hall will open door 2 (so that door 2 becomes door B and door 3 becomes door C). Similarly, if the prize is behind door 2, Monty Hall will open door 3. For these two cases Monty Hall had no choice.

However, what if the prize is actually behind door 1. Now what is Monty going to do? Maybe he always shows door 2 in this case. Or maybe he always shows door 3. Or maybe he uses some complicated secret algorithm for deciding which door to open. In any case the contestant has no knowledge as to how Monty will behave in this situation and so must regard Monty's two possibilities as equally likely. Hence, we may as well assume that Monty tosses a fair coin: if it comes up heads he opens door 2 while if it comes up tails he opens door 3. This random coin toss is independent of P. Therefore, our sample space consists of six points as shown in the table below.

If the contestant switches, then success will correspond to the four sample points (H,2), (H,3), (T,2) and (T,3). Hence the probability of success is $4/6 = 2/3$. On the other hand, if the contestant holds onto door A, then success will correspond to sample points (H,1) and (T,1) and the success probability will be only $1/3$.

Though they were unable to find any flaw in this line of argument, they were left quite puzzled since their intuition had failed them so miserably.

3 The Other Person's Envelope is Greener Paradox

Here is another paradox having to do with switching from one choice to another.

Two envelpes each contain an IOU for a specified amount of gold. One envelope is given to Ali and the other to Baba and they are told that the IOU in one envelope is worth twice as much as the other. However, neither knows who has the larger prize. Before anyone has opened their envelope, Ali is asked if she would like to trade her envelope with Baba. She reasons as follows. With 50 percent probability Baba's envelope contains half as much as mine and with 50 percent probability it contains twice as much. Hence, its expected value is $1/2(1/2) + 1/2(2) = 1.25$ which is 25 percent greater than what I already have and so yes it would be good to switch. Of course, Baba is presented with the same opportunity and reasons in the same way to conclude that he too would like to switch. So they switch and each thinks that his/her net worth just went up by 25 percent. Of course, since neither has yet openned any envelope, this process can be repeated and so again they switch. Now they are back with their original envelopes and yet they think that their fortune has increased 25 percent twice. They could continue this process ad infinitum and watch their expected worth zoom off to infinity.

Clearly, something is wrong with the above reasoning but where is the mistake? This paradox is quite puzzling until one darefully writes down a probabilistic model that describes the situation. Here is one possible model. Let X_0 denote the smaller amount of money between the two envelopes. This is a random variable taking values in the positive reals, but we (and more importantly, Ali and Baba) know nothing about its distribution. Let X_1 denote the larger amout of money so that $X_1 = 2X_0$. To select one of the two envelopes at random and give it to Ali means that we toss a fair coin and deliver to Ali either the envelope containing X_0 or the one containing X_1 depending on whether heads or tails appears. Mathematically, this means that we have another random variable N independent of X_0 (and hence of X_1) and taking values 0 and 1 with probability 1/2. The envelope that Ali receives contains $Y = X_N$ and the envelope that Baba receives contains $Z = X_{1-N}$.

Ali's reasoning about Baba's envelope starts by conditioning on the two possible values of N and bearing in mind that on the event $\{N = 0\}$, $Z = 2Y$ and on the event $\{N = 1\}$, $Z = Y/2$. Hence,

$$E[Z] = \frac{1}{2}E[2Y|N = 0] + \frac{1}{2}E[\frac{1}{2}Y|N = 1]. \tag{1}$$

At this point she mistakenly assumes that Y and N are independent and continues her argument as follows:

$$E[Z] = \frac{1}{2}E[2Y] + \frac{1}{2}E[\frac{1}{2}Y] = \frac{5}{4}E[Y].$$

Of course, the correct way to complete the analysis is to first note that

$$E[2Y|N=0] = E[2X_0|N=0] = 2E[X_0]$$

and

$$E[\frac{1}{2}Y|N=1] = E[\frac{1}{2}X_1|N=1] = \frac{1}{2}E[X_1] = E[X_0].$$

Then, substituting these into (1) we see that

$$E[Z] = \frac{1}{2}2E[X_0] + \frac{1}{2}E[X_0] = \frac{3}{2}E[X_0] = E[Y].$$

4 Another Switching Paradox

Here is another paradox which bears some similarity to the previous one. Suppose that two envelopes each contain an IOU for a specified amount of gold. You are asked to pick one envelope and open it up. Let E_1 denote the amount of gold specified in the envelope you opened and let E_2 denote the (unknown) amount specified in the other envelope. After opening it, you are offered the chance to switch your chosen envelope for the other one but you must make this choice with no knowledge concerning what is contained in the other envelope. The question is, can you find a strategy for which your chance of ending up with the better envelope is more than one half. At first blush, this would appear to be impossible.

But consider the following strategy. Suppose you set an apriori "satisfaction" tolerance of say 1.0 (gram). If E_1 is greater than 1.0, then keep the original amount, but if it is less switch to the other envelope.

Use a random number generator to generate a mean zero variance one normal random variable X and let $Y = e^X$. Your strategy will be to keep your original envelope if it shows more than Y and switch if it shows less than Y. Let's look at when ...