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The Earth Is

An Analysis of a Sunset Photo

Can a photo of the sunset over Lake Michigan reveal the shape of our planet?

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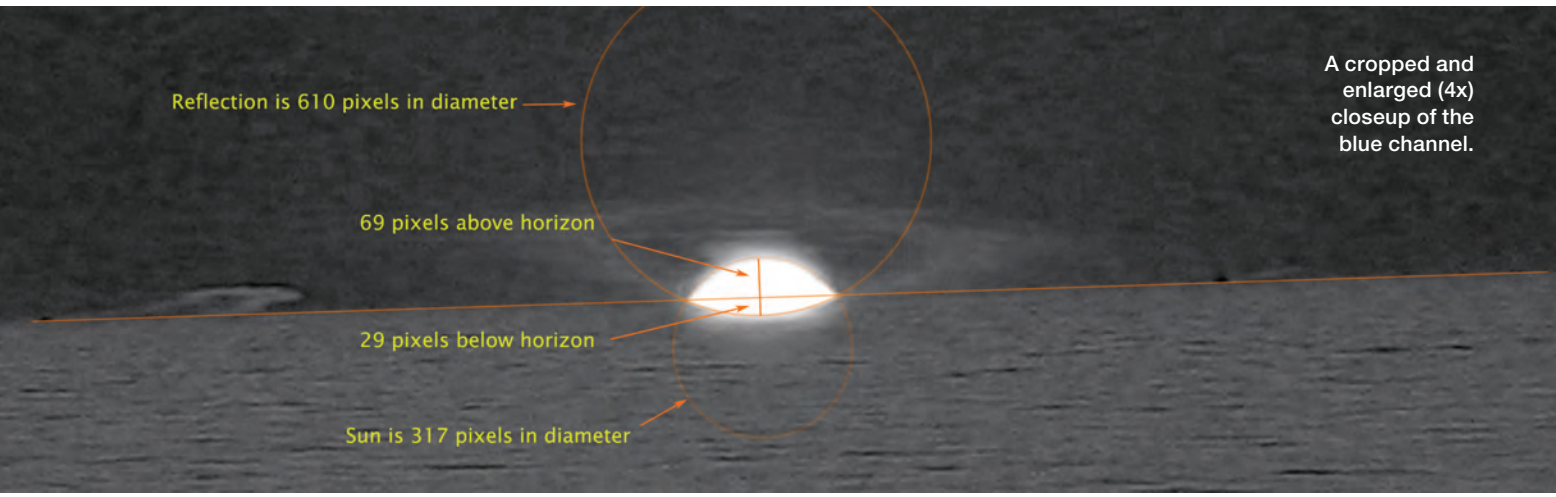
A sunset over Lake Michigan on July 5, 2008.
Photo by Robert J. Vanderbei

Not Flat



A cropped and enlarged (4x) closeup of the blue channel.

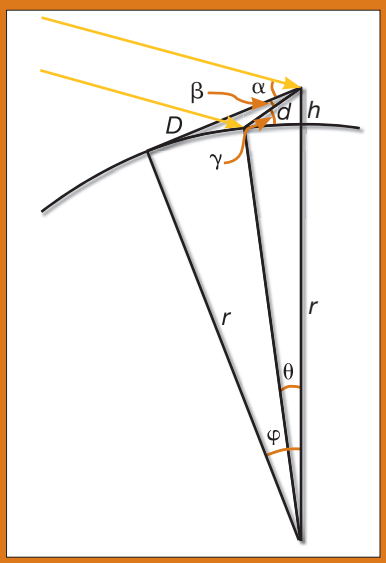
There are several well-known ways to prove to yourself that the Earth is round—by looking at NASA images, watching ships disappear over the horizon, or noting the presence of time zones, which wouldn't be possible without the longitudinal aspect of the planet. But did you know that you could reach the same conclusion simply by observing a sunset or sunrise over water on a calm, clear day? Moreover, from a photograph, you can even estimate, albeit rather crudely, the diameter of the Earth.

If the Earth were flat, the water calm, and the air clear, the Sun would make a perfect mirror image reflection on the water. The sunset image shown on the opening spread was taken over Lake Michigan on July 5, 2008. The cropped close-up above shows a reflection that is clearly foreshortened. While there may be many optical effects at play, such foreshortening can only be accounted for by the curvature of the Earth.

Here, we show how to measure the diameter of the Earth from a picture like this. We also discuss various complicating issues, the most prominent being the effect of waves roughening up the surface.

Foreshortening: the ideal case

We start by ignoring all complicating effects, including rough water and atmospheric effects such as refraction.



The paths of two sun rays coming from the upper limb of the sun. One ray bounces off the water and into the camera; the other goes straight to the camera.

The diagram above shows the relevant geometry. In that diagram,

- r represents the radius of the Earth,
- h represents the camera's height above the water,
- D is the distance to the horizon as seen from height h,
- φ is the corresponding angular measure,
- d is the distance to the point on the water corresponding to the bottom-most point of the reflection.
- θ is the corresponding angular measure,
- α is the angle from the horizon to the top of the sun as seen in the photograph,
- β is the angle from the horizon down to the bottom of the reflection as seen in the photograph, and
- γ is complementary to the angle of incidence for rays from the top-most part of the sun as they reflect off the water.

Of these nine quantities, three represent data that we can measure. For the opening photograph and cropped close-up, height h was not carefully recorded but is estimated to be about 7 feet above the level of the water. Angles α and β can be measured from the image in the cropped close-up and by drawing on the fact that the angular size of the sun is about 0.5 degrees.

The resolution isn't very good. However, based on the (4×) magnified close-up image, we can estimate that the diameter of the sun is 317 pixels, the diameter of the reflection is about 610 pixels, and the length of their common sector is 262 pixels. From these three measurements and the simple application of the Pythagorean theorem, we

calculate that the top of the sun is 69 pixels above the horizon, and the bottom of the reflection is 29 pixels below the horizon. (It is more accurate to compute these latter two distances from the first three measurements than to measure them directly.)

From these measurements, we gather that

$$\alpha = (69/317) \times 0.5° = 0.001907 \text{ rad}$$
$$\beta = (29/317) \times 0.5° = 0.000814 \text{ rad.}$$

There are six unknown parameters, including, of course, the one we are interested in, r. Fortunately, there are also six independent equations. They are:

$$\theta + \gamma = \varphi + \beta$$
$$\alpha = 2\gamma - \beta$$
$$r\cos(\theta) + d\sin(\theta + \gamma) = r + h$$
$$r\sin(\theta) - d\cos(\theta + \gamma) = 0$$
$$r\cos(\varphi) + D\sin(\varphi) = r + h$$
$$r\sin(\varphi) - D\cos(\varphi) = 0.$$

To see where the first equation comes from, imagine standing at the point on the water where the top-most rays of the sun are striking the water. Look straight up, perpendicular to the water surface. This is an angle of θ, counterclockwise from vertical in the diagram. Now look parallel to the water surface in the direction of the person standing on the shore. In the diagram, that is a clockwise rotation of 90 degrees. We are measuring angles from vertical using the standard convention that angles increase in the counterclockwise direction.

Hence, we are now gazing in the direction $\theta - 90$. Now tilt your gaze upward by γ degrees and you are looking straight at the camera. In other words, the angle of the segment of length d in the diagram is tilted at an angle of $\theta - 90 + \gamma$ from the vertical. From the perspective of the person holding the camera, the reversed direction, which is to say the direction looking from the camera to that point d miles away on the water, has a direction 180 degrees opposite. Hence, that direction is $\theta + 90 + \gamma$.

Now, if the camera person tilts his/her gaze up by β degrees, then the "look" direction is parallel to the segment of length D miles that touches the horizon tangentially off in the distance. Hence, this direction is $\theta + 90 + \gamma - \beta$. Because this direction is tangential to the water, it is clear that it can also be expressed as $\varphi + 90$. From this, we get the first of the six equations.

To arrive at the second equation, imagine the observer looking out at the horizon. Gazing downward by β degrees, the observer is now looking at the bottom-most reflection point of the sun. This visual ray strikes the water at an angle of γ. Hence, if the continuing ray were deviated upward by γ degrees, it would become a grazing path. Tilt it upward another γ degrees and it becomes parallel to the incoming light rays. That is, it becomes a ray having an angle of α degrees relative to the horizon.

The middle pair simply expresses the fact that the vector from the center of the Earth to the camera can be decomposed into a sum of two vectors: one from the center of the Earth to the horizon and the other from the horizon to the camera.

The last pair is similar. It decomposes the same vector into the path that goes from the center of the Earth to the bottom-most reflection point on the water to the camera.

What we have is a nonlinear system of six equations in six unknowns. One could use Newton's method to solve these equations with great accuracy. However, there is significant error on the measured quantities, so high precision seems unnecessary. Fortunately, for small angles, we can find an explicit approximate solution. One can show that

$$\varphi = \sqrt{\beta^2 + \left(\frac{\alpha-\beta}{2}\right)^2} - \beta + \frac{\alpha-\beta}{2}$$

and that $r = 2h/\varphi^2$.

For $\beta < \alpha$, we have $\varphi > 0$ and therefore $r < \infty$. But, if $\beta = \alpha$, then $\varphi = 0$ and r becomes infinite.

Plugging in the numbers

If we use

$$\alpha = (69/317) \times 0.5° = 0.001907 \text{ rad}$$
$$\beta = (29/317) \times 0.5° = 0.000814 \text{ rad,}$$

we find that $r = 5215$ miles, $d = 0.868$ miles and $D = 3.72$ miles.

The effect of waves

Normally, waves wreak havoc on sunset reflections. Perhaps the opening picture of calm water is a rarity. Certainly Lake Michigan and the other Great Lakes are calmer than the oceans. To analyze the effect of waves on the water, we assume for simplicity (but incorrectly) that the Earth is flat. The correct spherical case is similar. The diagram on the left shows the relevant geometry. We assume here that the waves are parallel to the line of sight so that a two-dimensional argument suffices for the moment.

Let $y = A \sin kx$ denote the surface of the wave. We shall consider the more general case later. We also assume that a point light source, such as light coming from one specific point on the sun, is "infinitely" far away off the right-hand

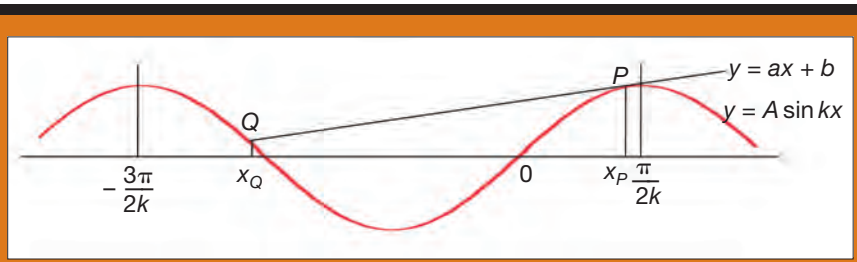


Diagram illustrating the paths of a sun ray that is coming from the right at a certain inclination, skimming over the right-hand wave, and then hitting the left-hand wave. Observers off the left-hand side of the page could see reflections of the light originating from the crest of the left-hand wave to point *Q* and from the point *P* up to the crest of the right-hand wave.

side of the diagram. This implies that its rays of light are parallel to each other. The diagram depicts just one such ray having an "inclination" given by a relative to horizontal.

The path of the light ray is given by $y = ax + b$.

We are interested in the ray that just touches the right-hand wave. The point P at which the ray and the wave touch and the constant b are determined by the equations $y' = a$ and $y' = Ak \cos kx$.

Tall waves

Assume that the maximum slope of the waves, Ak, is greater than the angle a of the sun ray. That is, assume that

$$\frac{a}{Ak} \leq 1 \ . \qquad (1)$$

We can use this small angle approximation again on trig functions (this time arccosine) to get that

$$x_P \approx \frac{1}{k}\left(\frac{\pi}{2} - \frac{a}{Ak}\right) \ .$$

Again, using small angle formulae, we find that

$$b = y_P - ax_P$$

$$\approx A\left(1 - \frac{\pi}{2}\frac{a}{Ak} + \frac{1}{2}\left(\frac{a}{Ak}\right)^2\right) \ .$$

Now that we know the formula for the light ray, we can figure out where Q is. In particular, we compute x_Q by equating the equation for the light ray and the equation for the wave surface:

$$ax_Q + b = A \sin kx_Q$$

$$= A \cos\left(kx_Q + \frac{3\pi}{2}\right)$$

$$\approx A\left(1 - \frac{1}{2}\left(kx_Q + \frac{3\pi}{2}\right)^2\right) \ .$$

Solving the resulting quadratic formula and invoking the small angle assumption Eq. (1), we see that

$$x_Q \approx \frac{1}{k}\sqrt{2\pi\frac{a}{Ak}} - \frac{3\pi}{2k} \ .$$

The angle at which the ray of light reflects off from the wave can be found easily from the slope at the point of incidence. So, in particular, we have

$$y'(x_Q) = Ak \cos kx_Q$$

$$= Ak \cos\left(\sqrt{2\pi\frac{a}{Ak}} - \frac{3\pi}{2}\right)$$

$$= -Ak \sin\sqrt{2\pi\frac{a}{Ak}}$$

$$\approx -Ak\sqrt{2\pi\frac{a}{Ak}}$$

$$= -\sqrt{2\pi aAk}$$

and $y'(x_P) = a$.

From these two slopes, it is easy to see that the reflection angle measured upward from horizontal can vary from $-a$ to $a + 2(2\pi aAk)^{1/2}$. Of course,

One of the most common historical errors people make is to assume that medieval civilizations did not know that the Earth was round. As this 15th century map shows, medieval civilizations were aware of the planet's spherical nature. In fact, the Earth's roundness had been known since at least 500 B.C., when Pythagoras came to that conclusion based on the way that the altitudes of stars varied at different places on Earth.

a downward angled reflection will be intercepted by the next wave (as is the case in the figure). Hence, the range of upward reflection angles (measured positively) varies from essentially zero to

$$\text{max angle} = a + 2\sqrt{2\pi aAk}\,.$$

Clearly, the maximum angle is greater than a. This means that the reflection should appear lengthened, not shortened. Furthermore, because of Eq. (1), the term $(2\pi aAk)^{1/2}$ is greater than a. (The negative slope at Q is greater in magnitude than the positive slope at P.) Hence, the points of greatest lengthening should appear more than three times the angle of a straight reflection. This would seem to imply that a picture such as the one shown would not be possible. We have made two assumptions, both of which are suspect.

Small waves

In the previous section, we assumed that the waves were tall (relative to the angle a of the incoming beam). Suppose that the waves are small rather than tall. That is, suppose that the inequality in Eq. (1) is reversed. In this case, the incoming light beam illuminates all parts of the wave, even down to the bottom of the troughs. The reflection coming off at the steepest angle is the one that hits the wave where its angle is steepest—i.e., where it crosses the horizontal axis in the diagram on p. 37.

If the wave were flat, then the reflection angle would be a. But, the wave tips the reflection angle by twice its own slope. Its maximum slope is Ak. Hence, the greatest possible angle of the reflected beam is $a + 2Ak$. If Ak is much smaller than a, then this reflected angle well approximates a itself.

Waves not parallel to the line of sight

The second suspicious assumption is that the waves are parallel to the line of sight. Certainly close to shore, this should be approximately correct and the picture bears it out. Of course, the surface of the lake is a superposition of many waves. The principal component is the wave generated by wind over the entire surface of the lake. (Lake Michigan is about 100 miles wide and several hundred miles long). This is the wave that arrives at the shore as "surf." It is well known that waves angle toward shore as the depth of the lake decreases toward shore. Hence, it is not surprising that there is always some "surf."

But, further out, the principal wave could be going in almost any direction, perhaps even perpendicular to shore. In that case, the reflection would be "blurred" to the left and right but not up and down. It is not possible to determine independently from the photograph the orientation of the principal wind-generated wave.

There is also the multitude of other waves generated by local effects, mainly by boats going by. Close inspection of the picture reveals that there were only two boats out near the horizon when the picture was taken. Both were heading toward the reflection point rather than away from it. Therefore, the water where we see the reflection was probably undisturbed for several minutes prior to the picture being taken; this probably accounts significantly for the quality of the reflection.

The horizon anomaly

There is some ambiguity as to how one draws the horizon. One can draw it using the apparent horizon, as seen close to the sun or using a pair of points much farther away. In the cropped close-up on p. 36, the line drawn corresponds to the latter case, in which the reference points are well to either side of the sun. This line is about six pixels (1.5 pixels in the original unenlarged photograph) higher than what one gets with a close drawing. What can account for this strange anomaly?

It seems that the most sensible explanation is that the small waves out near the horizon were probably oriented close to perpendicular to the line of sight. This would certainly be consistent with our earlier conclusion that the only way for the reflection not to be extended downward is for it to be extended side to side.

Atmospheric effects

The air was obviously uncommonly clear when the picture was taken. There seem to be no mirage effects from low-level temperature inversions. Also, one must consider the effect of refraction as the sunlight came in through the atmosphere. Had there been no atmospheric refraction, we would see that the sun had already set when the photograph was taken. How does refraction affect the image? The answer to this question is: very little.

The only difference between the unreflected rays and the reflected ones is that the reflected ones came through the atmosphere a few miles to the west of the unreflected ones. Both sets are refracted by almost the same amount. The difference in the angle, which is determined by, say, four miles on 4,000 miles is such a small angle that the differential refraction is tiny. Absolute refraction is significant, but differential refraction is negligible.

The effect of saturated pixels

One could point out that the sun and its reflection have both saturated the pixels on the CCD, and maybe one needs to worry about bleeding. To check on this, the color picture was separated into its constituent red, green and blue components. The red and green components were saturated but the blue one was not. Taking measurements from the blue component, one essentially achieves the same results as before.

Although it is difficult to account for the effect of waves on the water, any foreshortening of the shadow is definitive proof that the Earth is not flat. And, the simple measurements one can make from the photograph provide an estimate of the Earth's radius that is in the same ballpark as the known value.

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