Measuring the Earth's Diameter from a Sunset Photo

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ABSTRACT

The Earth is not flat. We all know this. But, can you prove it? Believe it or not, one can measure the diameter of the Earth from a single photograph of a sunset such as the one shown in Figure 1. Take a close look at this picture. The Sun is setting. Above the horizon, we see a small remaining portion of the top of the Sun. But, we also see a reflection of this part of the Sun in the water. If the Earth were flat (and the water calm), then the reflection would be a simple mirror image of what we see above the horizon. But, the reflected part is foreshortened a measurable amount. Such a foreshortening is proof that the Earth is not flat. But, what's more, by taking some simple measurements from the photo and knowing about how high above the surface of the Earth the photo was taken, we can actually compute the radius of the Earth.

Assuming that I was standing about 7 feet above the surface of Lake Michigan when I took this photograph and assuming that we already know that the diameter of the Sun is about $1/2°$, we can compute that the radius of the Earth is somewhere between 4000 and 9100 miles (depending on where I place the horizon line). Of course, we know the correct answer is about 4000. There is clearly a tendency for this measurement to be too large. The main reason is the fact that waves on Lake Michigan make the reflection less foreshortened than it should be. In fact most sunset pictures show the reflected part not foreshortened at all but rather extended in a big blur that often extends right up to the shoreline. Such extension is caused by waves. If there are waves and undulations, then the water does not behave like a flat, or spherical, mirror. In such cases, there is reflected light all over the place. So, I was quite lucky to get this sunset picture. There was no wind and the body of water was a lake, not an ocean, and as such did not have "surf". Anyway, I can attest, and the picture confirms, that the lake was quite calm indeed. So, the computation actually gives an upper bound on the radius. And, from this picture, we can see that the upper bound is most definitely finite and not far from the correct value.

Higher resolution photographs taken from a higher, more precisely measured, height could yield a more accurate measurement. But, this was the photo I took. I wasn't expecting

Fig. 1.— A sunset over Lake Michigan on July 5, 2008.

to use it for this measurement. Also, I didn't record how high above the water I was standing. All I know is that I was close to the shoreline. I am just guessing that this put me maybe about a foot above water level and since my eye level is slightly less that 6 feet, a guess of 7 feet seems about right.

Figure 2 shows the relevant geometry. In that diagram,

- r represents the radius of the Earth,
- h represents the camera's height above the water,
- D is the distance to the horizon as seen from height h,
- φ is the corresponding angular measure,
- d is the distance to the point on the water corresponding to the bottom-most point of the reflection.
- θ is the corresponding angular measure,
- α is the angle from the horizon to the top of the sun as seen in the photograph,
- β is the angle from the horizon down to the bottom of the reflection as seen in the photograph, and
- γ is complementary to the angle of incidence for rays from the top-most part of the Sun as they reflect off the water.

Of these nine quantities, three represent data that we can measure. As already mentioned, we take the height h to be about 7 feet. We can measure the angles α and β using a closeup version of the sunset photo and the fact that the angular size of the Sun is about 0.5 degrees. See Figure 3. The resolution isn't very good but based on the ($4\times$) magnified closeup image we can estimate that the diameter of the Sun is 317 pixels, the top of the Sun is 66 pixels above the horizon, and the bottom of the reflection is 34 pixels below the horizon. From these measurements, we get that

$$\begin{aligned}\alpha &= (66/317) \times 0.5 \text{ degrees} \\ \beta &= (34/317) \times 0.5 \text{ degrees.}\end{aligned}$$

Hence, there are six unknowns. Of course, none of these ruminations do us any good if we can't figure out these quantities (and most importantly r). Fortunately, there are also

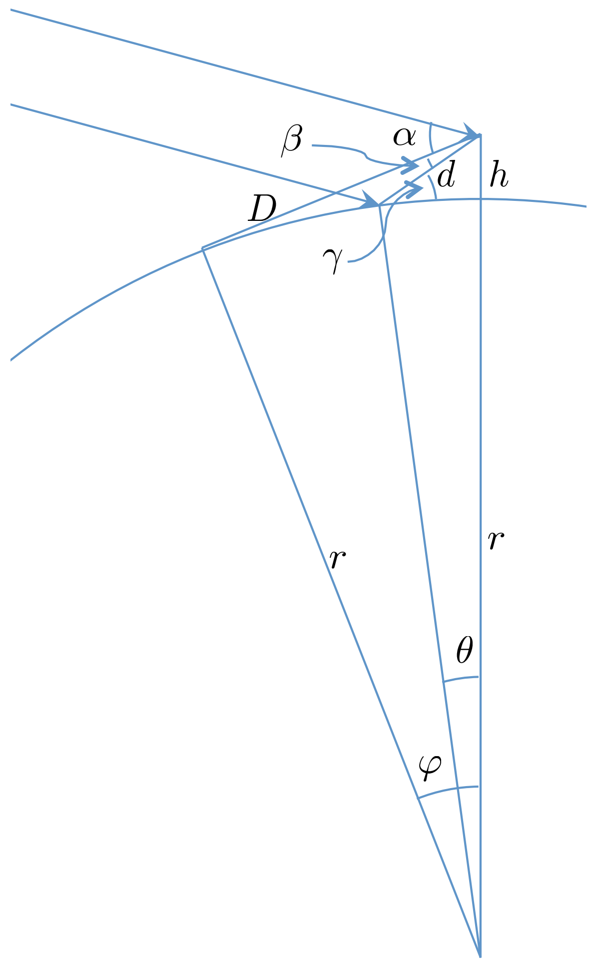

Fig. 2.— Diagram illustrating the paths of two Sun rays coming from the upper limb of the Sun. One ray bounces off the water and into the camera. The other goes straight to the camera.

six independent equations. They are

$$\begin{aligned}
\theta + \gamma &= \varphi + \beta \\
\alpha &= 2\gamma - \beta \\
r\cos(\theta) + d\sin(\theta + \gamma) &= r + h \\
r\sin(\theta) - d\cos(\theta + \gamma) &= 0 \\
r\cos(\varphi) + D\sin(\varphi) &= r + h \\
r\sin(\varphi) - D\cos(\varphi) &= 0.
\end{aligned}$$

Perhaps these equations need some explanation. The first equation is the most subtle (meaning it is the last one I figured out). Imagine standing at the point on the water where the top-most rays of the Sun are striking the water. Look straight up (perpendicular to the water surface). This is an angle of θ counterclockwise from vertical in the figure. Now look parallel to the water surface in the direction of the person standing on the shore. In the figure, that is a clockwise rotation of 90 degrees. Since measuring angles in the figure from vertical with acute counterclockwise angles being positive, we are now gazing in the direction $\theta - 90$. Now tilt the gaze upward by γ degrees and you are looking straight at the camera. In other words, the angle of the segment of length d in the figure is tilted at an angle of $\theta - 90 + \gamma$ from the vertical. From the perspective of the person holding the camera, the reversed direction, which is to say the direction looking from the camera to that point d miles away on the water, has a direction 180 degrees opposite. Hence that direction is $\theta + 90 + \gamma$. Now, if the camera person tilts his/her gaze up by β degrees, then the "look" direction is parallel to the segment of length D miles that touches the horizon tangentially off in the distance. Hence, this direction is $\theta + 90 + \gamma - \beta$ (don't forget that clockwise rotations are going in the negative direction, hence we subtract). Because this direction is tangential to

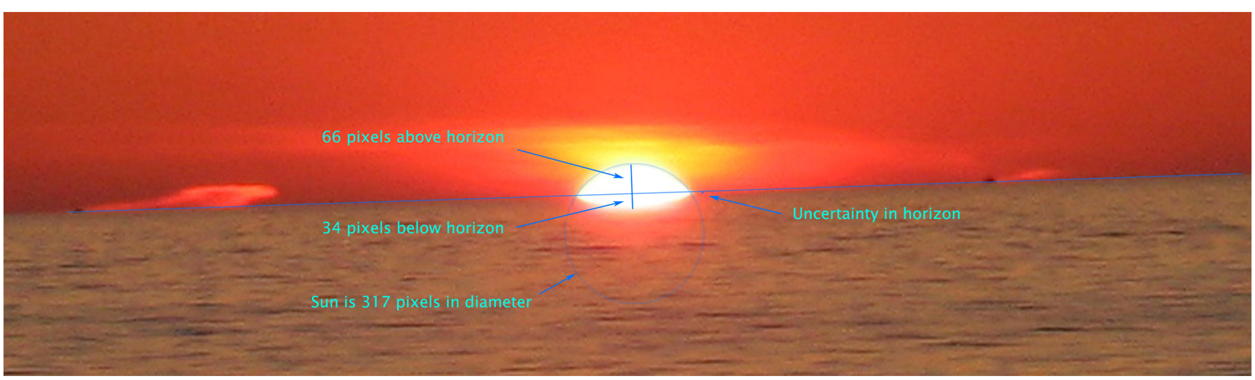

Fig. 3.— A closeup of the original sunset picture showing the measurements made.

the water, it is clear that it can also be expressed as $\varphi + 90$. Equating, we get the first of the six equations.

To see the second equation, imagine the observer looking out at the horizon. Gazing downward by β degrees, the observer is now looking at the bottom-most reflection point of the Sun. This visual ray strikes the water at an angle of γ. Hence, if the continuing ray were deviated upward by γ degrees it would become a grazing path. Tilting it upward another γ degrees and it becomes parallel to the incoming light rays. That is, it becomes a ray having an angle of α degrees relative to the horizon.

The middle pair simply express the fact that the vector from the center of the Earth to the camera can be decomposed into a sum of two vectors: one from the center of the Earth to the horizon and the other from the horizon to the camera.

The last pair is similar. It decomposes the same vector into the path that goes from the center of the Earth to the bottom-most reflection point on the water to the camera.

What we have is a nonlinear system of six equations in six unknowns. For small angles, we can find an explicit approximate solution. One can show that

$$\varphi = \sqrt{\beta^2 + \left(\frac{\alpha - \beta}{2}\right)^2} - \beta + \frac{\alpha - \beta}{2}$$

and that

$$r = \frac{2h}{\varphi^2}$$

For $\beta < \alpha$, we have $\varphi > 0$ and therefore $r < \infty$. But, if $\beta = \alpha$, then $\varphi = 0$ and r becomes infinite.

After enlarging the image by a factor of four, we measure the diameter of the Sun to be about 317 pixels. It is hard to get an accurate measurement of the angles α and β. The difficulty is determining where to draw the horizon line. Mysteriously, the horizon is not flat. Instead it dips down a few pixels in the vicinity of the Sun. If we use widely separated endpoints we get a horizon that is higher than if we use endpoints close to the edges of the Sun. Using the higher horizon, we get

$$\begin{aligned} \alpha &= (66/317) \times 0.5 \text{ degrees} \\ \beta &= (34/317) \times 0.5 \text{ degrees.} \end{aligned}$$

The lower estimate of the horizon is 6 pixels lower and so in this case we get

$$\begin{aligned} \alpha &= (72/317) \times 0.5 \text{ degrees} \\ \beta &= (28/317) \times 0.5 \text{ degrees.} \end{aligned}$$

Using the higher estimate of the horizon, we find that

$$\begin{aligned} r &= 9100 \text{ miles} \\ d &= 0.90 \text{ miles} \\ D &= 4.92 \text{ miles.} \end{aligned}$$

With the lower estimate, we get

$$\begin{aligned} r &= 4000 \text{ miles} \\ d &= 0.86 \text{ miles} \\ D &= 3.25 \text{ miles.} \end{aligned}$$

Next time I have the opportunity to take a sunset or sunrise picture over a body of water on a nice low-humidity day with no wind and no surf, I will try to take higher-resolution pictures from greater height above the water. In the meantime, it seems quite amazing to me that I was able to take a simple measurement based only on some angles and a single height measurement (my standing height above the water) and compute a reasonable estimate for the radius of the Earth.