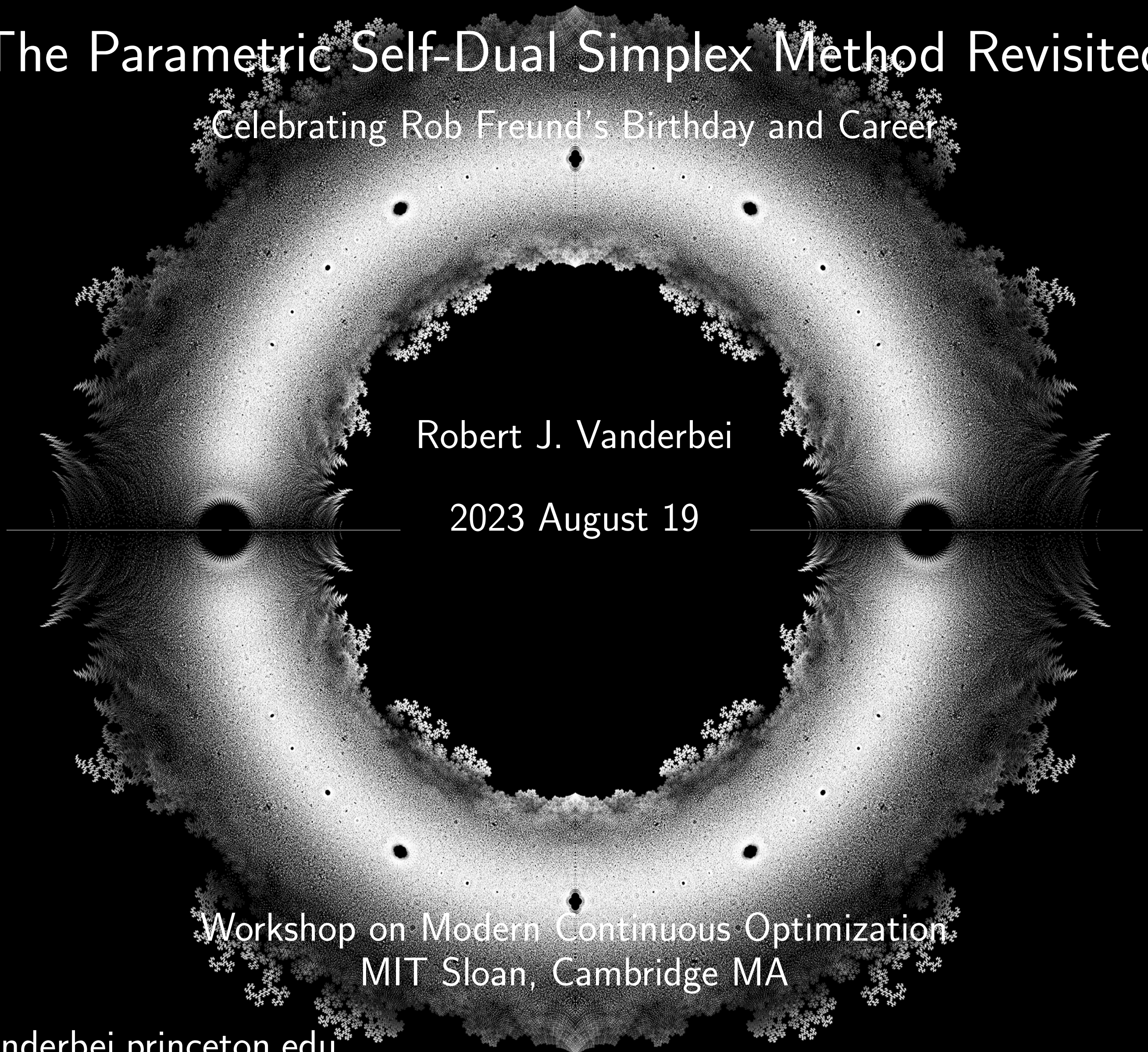
# The Parametric Self-Dual Simplex Method Revisited

Celebrating Rob Freund's Birthday and Career



Robert J. Vanderbei

2023 August 19

Workshop on Modern Continuous Optimization
MIT Sloan, Cambridge MA

http://vanderbei.princeton.edu

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To Rob:

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- I'm a glider pilot. I gave Rob three rides on Apr. 19, 1993. The third one was really really good–released from tow at 4000 feet and climbed to 7600 feet.
- Rob and I have been friends for 35 years. But, we've only co-authored one paper together and that was 31 years ago...

Prior Reduced Fill-in in Solving Equations in Interior Point Algorithms

John Birge, Robert Freund, Robert Vanderbei

Operations Research Letters, 1992

# Linear Optimization in "Symmetric Form"

Here's a *primal* problem in "standard" (aka inequality) form:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

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**Weak Duality** If $x$ is primal feasible and $y$ is dual feasible, then $c^T x \leq y^T Ax \leq y^T b$.

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**Duality** The dual of the dual is the primal.

**Weak Duality** If $x$ is primal feasible and $y$ is dual feasible, then $c^T x \leq y^T A x \leq y^T b$.

**Strong Duality** If $x$ is optimal for the primal, then there exists a dual-feasible $y$ such that

$$c^T x = b^T y.$$

# An Example

## Primal Problem:

$$
\begin{array}{lrcrcrcr}
\text{maximize} & -3x_1 & + & 11x_2 & + & 2x_3 & & \\
\text{subj. to} & -x_1 & + & 3x_2 & & & \leq & 5 \\
& 3x_1 & + & 3x_2 & & & \leq & 4 \\
& & & 3x_2 & + & 2x_3 & \leq & 6 \\
& -3x_1 & & & - & 5x_3 & \leq & -4 \\
& & & & & x_1,\ x_2,\ x_3 & \geq & 0
\end{array}
$$

### Written in *Dictionary Form*:

$$
\begin{array}{rcrcrcrcr}
\zeta & = & & & -3x_1 & + & 11\,x_2 & + & 2\,x_3 \\
\hline
w_1 & = & 5 & + & x_1 & - & 3x_2 & & \\
w_2 & = & 4 & - & 3x_1 & - & 3x_2 & & \\
w_3 & = & 6 & & & - & 3x_2 & - & 2x_3 \\
w_4 & = & -4 & + & 3x_1 & & & + & 5x_3
\end{array}
$$

### *Dictionary Solution:*

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,$$
$$w_1 = 5,\ w_2 = 4,\ w_3 = 6,\ w_4 = -4$$

## Dual Problem:

$$
\begin{array}{lrcrcrcrcr}
-\text{maximize} & -5y_1 & - & 4y_2 & - & 6y_3 & + & 4y_4 & & \\
\text{subj. to} & y_1 & - & 3y_2 & & & + & 3y_4 & \leq & 3 \\
& -3y_1 & - & 3y_2 & - & 3y_3 & & & \leq & -11 \\
& & & & - & 2y_3 & + & 5y_4 & \leq & -2 \\
& & & & & & & y_1,\ y_2,\ y_3,\ y_4 & \geq & 0
\end{array}
$$

### Written in *Dictionary Form*:

$$
\begin{array}{rcrcrcrcrcr}
-\xi & = & & & -5y_1 & - & 4y_2 & - & 6y_3 & + & 4\,y_4 \\
\hline
z_1 & = & 3 & - & y_1 & + & 3y_2 & & & - & 3y_4 \\
z_2 & = & -11 & + & 3y_1 & + & 3y_2 & + & 3y_3 & & \\
z_3 & = & -2 & & & & & + & 2y_3 & - & 5y_4
\end{array}
$$

### *Dictionary Solution:*

$$y_1 = 0,\ y_2 = 0,\ y_3 = 0,\ y_4 = 0,$$
$$z_1 = 3,\ z_2 = -11\,,\ z_3 = -2$$

# An Example

## Primal Problem:

$$\begin{array}{lrcrcrcl} \text{maximize} & -3x_1 & + & 11x_2 & + & 2x_3 & & \\ \text{subj. to} & -x_1 & + & 3x_2 & & & \leq & 5 \\ & 3x_1 & + & 3x_2 & & & \leq & 4 \\ & & & 3x_2 & + & 2x_3 & \leq & 6 \\ & -3x_1 & & & - & 5x_3 & \leq & -4 \\ & & & & x_1, & x_2, \; x_3 & \geq & 0 \end{array}$$

## Dual Problem:

$$\begin{array}{lrcrcrcrcl} -\text{maximize} & -5y_1 & - & 4y_2 & - & 6y_3 & + & 4y_4 & & \\ \text{subj. to} & y_1 & - & 3y_2 & & & + & 3y_4 & \leq & 3 \\ & -3y_1 & - & 3y_2 & - & 3y_3 & & & \leq & -11 \\ & & & & - & 2y_3 & + & 5y_4 & \leq & -2 \\ & & & & & y_1, \; y_2, & y_3, & y_4 & \geq & 0 \end{array}$$

## Written in *Dictionary Form*:

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## *Dictionary Solution:*

$$x_1 = 0, \; x_2 = 0, \; x_3 = 0,$$
$$w_1 = 5, \; w_2 = 4, \; w_3 = 6, \; w_4 = -4$$

## *Dictionary Solution:*

$$y_1 = 0, \; y_2 = 0, \; y_3 = 0, \; y_4 = 0,$$
$$z_1 = 3, \; z_2 = -11, \; z_3 = -2$$

Note: Current "solution" is neither primal nor dual feasible.

# Parametric Self-Dual Simplex Method

Introduce a parameter $\mu$ and perturb:

*Primal Problem:*

$$
\begin{array}{rcrcrcrcrcr}
\zeta & = & & & & & -3x_1 & + & 11x_2 & + & 2x_3 \\
 & & & & & & -\mu x_1 & - & \mu x_2 & - & \mu x_3 \\
\hline
w_1 & = & 5 & + & \mu & + & x_1 & - & 3x_2 & & \\
w_2 & = & 4 & + & \mu & - & 3x_1 & - & 3x_2 & & \\
w_3 & = & 6 & + & \mu & & & - & 3x_2 & - & 2x_3 \\
w_4 & = & -4 & + & \mu & + & 3x_1 & & & + & 5x_3
\end{array}
$$

*Dual Problem:*

$$
\begin{array}{rcrcrcrcrcrcr}
-\xi & = & & & & & -5y_1 & - & 4y_2 & - & 6y_3 & + & 4y_4 \\
 & & & & & & -\mu y_1 & - & \mu y_2 & - & \mu y_3 & - & \mu y_4 \\
\hline
z_1 & = & 3 & + & \mu & - & y_1 & + & 3y_2 & & & - & 3y_4 \\
z_2 & = & -11 & + & \mu & + & 3y_1 & + & 3y_2 & + & 3y_3 & & \\
z_3 & = & -2 & + & \mu & & & & & + & 2y_3 & - & 5y_4
\end{array}
$$

Here's how the primal version looks in my online pivot tool:

maximize

$$\zeta = 0 + 0\,\mu + (-3)\,x_1 + 11\,x_2 + 2\,x_3$$
$$\quad + 0\,\mu + 0\,\mu^2 + (-1)\,\mu\,x_1 + (-1)\,\mu\,x_2 + (-1)\,\mu\,x_3$$

subject to:

$$
\begin{array}{rcrcrcrcrcrcr}
w_1 & = & 5 & + & 1\,\mu & - & -1\,x_1 & - & 3\,x_2 & - & 0\,x_3 \\
w_2 & = & 4 & + & 1\,\mu & - & 3\,x_1 & - & 3\,x_2 & - & 0\,x_3 \\
w_3 & = & 6 & + & 1\,\mu & - & 0\,x_1 & - & 3\,x_2 & - & 2\,x_3 \\
w_4 & = & -4 & + & 1\,\mu & - & -3\,x_1 & - & 0\,x_2 & - & -5\,x_3
\end{array}
$$

$$x_1,\ x_2,\ x_3,\quad w_1,\ w_2,\ w_3,\ w_4 \ \geq\ 0$$

$$11 \leq \mu \leq \infty$$

For $\mu \geq 11$, dictionary is optimal. $x_2$ is the *entering variable* and $w_2$ is the *leaving variable*.

# Before and After the First Pivot

$$
\begin{array}{llcl}
\text{maximize} & \zeta & = & 0 + 0\,\mu + (-3)\,x_1 + 11\,x_2 + 2\,x_3 \\
 & & & + 0\,\mu + 0\,\mu^2 + (-1)\,\mu\,x_1 + (-1)\,\mu\,x_2 + (-1)\,\mu\,x_3 \\
\text{subject to:} & w_1 & = & 5 + 1\,\mu - (-1)\,x_1 - 3\,x_2 - 0\,x_3 \\
 & w_2 & = & 4 + 1\,\mu - 3\,x_1 - 3\,x_2 - 0\,x_3 \\
 & w_3 & = & 6 + 1\,\mu - 0\,x_1 - 3\,x_2 - 2\,x_3 \\
 & w_4 & = & -4 + 1\,\mu - (-3)\,x_1 - 0\,x_2 - (-5)\,x_3
\end{array}
$$

$$x_1,\ x_2,\ x_3,\quad w_1,\ w_2,\ w_3,\ w_4\ \geq 0$$

$$11 \leq \mu \leq \infty$$

$$
\begin{array}{llcl}
\text{maximize} & \zeta & = & 44/3 + 11/3\,\mu + (-14)\,x_1 + (-11/3)\,w_2 + 2\,x_3 \\
 & & & + (-4/3)\,\mu + (-1/3)\,\mu^2 + 0\,\mu\,x_1 + 1/3\,\mu\,w_2 + (-1)\,\mu\,x_3 \\
\text{subject to:} & w_1 & = & 1 + 0\,\mu - (-4)\,x_1 - (-1)\,w_2 - 0\,x_3 \\
 & x_2 & = & 4/3 + 1/3\,\mu - 1\,x_1 - 1/3\,w_2 - 0\,x_3 \\
 & w_3 & = & 2 + 0\,\mu - (-3)\,x_1 - (-1)\,w_2 - 2\,x_3 \\
 & w_4 & = & -4 + 1\,\mu - (-3)\,x_1 - 0\,w_2 - (-5)\,x_3
\end{array}
$$

$$x_1,\ x_2,\ x_3,\quad w_1,\ w_2,\ w_3,\ w_4\ \geq 0$$

$$4 \leq \mu \leq 11$$

# Before and After the Second Pivot

$$\text{maximize} \quad \zeta = \frac{44}{3} + \frac{11}{3}\mu - 14 x_1 - \frac{11}{3} w_2 + 2 x_3 - \frac{4}{3}\mu - \frac{1}{3}\mu^2 + 0\mu x_1 + \frac{1}{3}\mu w_2 - 1\mu x_3$$

$$\begin{aligned}
\text{subject to:} \quad w_1 &= 1 + 0\mu - (-4) x_1 - (-1) w_2 - 0 x_3 \\
x_2 &= \frac{4}{3} + \frac{1}{3}\mu - 1 x_1 - \frac{1}{3} w_2 - 0 x_3 \\
w_3 &= 2 + 0\mu - (-3) x_1 - (-1) w_2 - 2 x_3 \\
w_4 &= -4 + 1\mu - (-3) x_1 - 0 w_2 - (-5) x_3
\end{aligned}$$

$$x_1,\ x_2,\ x_3,\quad w_1,\ w_2,\ w_3,\ w_4 \geq 0$$

$$4 \leq \mu \leq 11$$

$$\text{maximize} \quad \zeta = \frac{244}{15} + \frac{49}{15}\mu - \frac{76}{5} x_1 - \frac{11}{3} w_2 + \frac{2}{5} w_4 - \frac{32}{15}\mu - \frac{2}{15}\mu^2 + \frac{3}{5}\mu x_1 + \frac{1}{3}\mu w_2 - \frac{1}{5}\mu w_4$$

$$\begin{aligned}
\text{subject to:} \quad w_1 &= 1 + 0\mu - (-4) x_1 - (-1) w_2 - 0 w_4 \\
x_2 &= \frac{4}{3} + \frac{1}{3}\mu - 1 x_1 - \frac{1}{3} w_2 - 0 w_4 \\
w_3 &= \frac{2}{5} + \frac{2}{5}\mu - \left(-\frac{21}{5}\right) x_1 - (-1) w_2 - \frac{2}{5} w_4 \\
x_3 &= \frac{4}{5} - \frac{1}{5}\mu - \frac{3}{5} x_1 - 0 w_2 - \left(-\frac{1}{5}\right) w_4
\end{aligned}$$

$$x_1,\ w_2,\ w_4,\quad w_1,\ x_2,\ w_3,\ x_3 \geq 0$$

$$2 \leq \mu \leq 4$$

# Before and After the Third Pivot

$$
\begin{array}{llll}
\text{maximize} & \zeta & = & 244/15 + 49/15\,\mu + (-76/5)\,x_1 + (-11/3)\,w_2 + 2/5\,w_4 \\
& & & + (-32/15)\,\mu + (-2/15)\,\mu^2 + 3/5\,\mu\,x_1 + 1/3\,\mu\,w_2 + (-1/5)\,\mu\,w_4 \\
\text{subject to:} & w_1 & = & 1 + 0\,\mu - (-4)\,x_1 - (-1)\,w_2 - 0\,w_4 \\
& x_2 & = & 4/3 + 1/3\,\mu - 1\,x_1 - 1/3\,w_2 - 0\,w_4 \\
& w_3 & = & 2/5 + 2/5\,\mu - (-21/5)\,x_1 - (-1)\,w_2 - 2/5\,w_4 \\
& x_3 & = & 4/5 + (-1/5)\,\mu - 3/5\,x_1 - 0\,w_2 - (-1/5)\,w_4
\end{array}
$$

$$x_1,\ w_2,\ w_4,\quad w_1,\ x_2,\ w_3,\ x_3 \geq 0$$

$$2 \leq \mu \leq 4$$

$$
\begin{array}{llll}
\text{maximize} & \zeta & = & 50/3 + 11/3\,\mu + (-11)\,x_1 + (-8/3)\,w_2 + (-1)\,w_3 \\
& & & + (-7/3)\,\mu + (-1/3)\,\mu^2 + (-3/2)\,\mu\,x_1 + (-1/6)\,\mu\,w_2 + 1/2\,\mu\,w_3 \\
\text{subject to:} & w_1 & = & 1 + 0\,\mu - (-4)\,x_1 - (-1)\,w_2 - 0\,w_3 \\
& x_2 & = & 4/3 + 1/3\,\mu - 1\,x_1 - 1/3\,w_2 - 0\,w_3 \\
& w_4 & = & 1 + 1\,\mu - (-21/2)\,x_1 - (-5/2)\,w_2 - 5/2\,w_3 \\
& x_3 & = & 1 + 0\,\mu - (-3/2)\,x_1 - (-1/2)\,w_2 - 1/2\,w_3
\end{array}
$$

$$x_1,\ w_2,\ w_4,\quad w_1,\ x_2,\ w_3,\ x_3 \geq 0$$

$$-1 \leq \mu \leq 2$$

We're done! It's *optimal*.

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Okay, there are only 6 items in the list. SORRY.

# Worst Case is Exponential

Using only $\pm 1$'s for the initial perturbation coefficients, the parametric self-dual simplex method used on the Klee-Minty problem takes an exponential number of pivots.

Here it is with $n = 4$...

**Current Dictionary**

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| maximize | ζ | = | 0 | | + | 0 | μ | + | 8 | | x1 | + | 4 | | x2 | + | 2 | | x3 | + | 1 | | x4 |
| | | + | 0 | μ | + | 0 | μ² | + | -1 | μ | x1 | + | -1 | μ | x2 | + | -1 | μ | x3 | + | -1 | μ | x4 |
| subject to: | w1 | = | 1 | | + | 1 | μ | - | 1 | | x1 | - | 0 | | x2 | - | 0 | | x3 | - | 0 | | x4 |
| | w2 | = | 100 | | + | 1 | μ | - | 4 | | x1 | - | 1 | | x2 | - | 0 | | x3 | - | 0 | | x4 |
| | w3 | = | 10000 | | + | 1 | μ | - | 8 | | x1 | - | 4 | | x2 | - | 1 | | x3 | - | 0 | | x4 |
| | w4 | = | 1000000 | | + | 1 | μ | - | 16 | | x1 | - | 8 | | x2 | - | 4 | | x3 | - | 1 | | x4 |

$x_1, x_2, x_3, x_4, \quad w_1, w_2, w_3, w_4 \geq 0$

$8 \leq \mu \leq \infty$

The problem, as shown, takes $2^n - 1 = 15$ pivots.

And, as usual with the Klee-Minty problem we can change the parameter coefficients so that $x_4$ is the first entering variable and the algorithm converges in just *one pivot*.

# Expected Number of Pivots

Thought experiment:

- $\mu$ starts at $\infty$.
- In reducing $\mu$, there are $n + m$ barriers.
- At each iteration, one barrier is passed—the others move about "randomly".
- To get $\mu$ to zero, we must on average pass half the barriers.
- Therefore, on average the algorithm should take $(m + n)/2$ iterations.

# Real-World Data

| Name | $m$ | $n$ | iters | Name | $m$ | $n$ | iters |
|---|---|---|---|---|---|---|---|
| 25fv47 | 777 | 1545 | 5089 | nesm | 646 | 2740 | 5829 |
| 80bau3b | 2021 | 9195 | 10514 | recipe | 74 | 136 | 80 |
| adlittle | 53 | 96 | 141 | sc105 | 104 | 103 | 92 |
| afiro | 25 | 32 | 16 | sc205 | 203 | 202 | 191 |
| agg2 | 481 | 301 | 204 | sc50a | 49 | 48 | 46 |
| agg3 | 481 | 301 | 193 | sc50b | 48 | 48 | 53 |
| bandm | 224 | 379 | 1139 | scagr25 | 347 | 499 | 1336 |
| beaconfd | 111 | 172 | 113 | scagr7 | 95 | 139 | 339 |
| blend | 72 | 83 | 117 | scfxm1 | 282 | 439 | 531 |
| bnl1 | 564 | 1113 | 2580 | scfxm2 | 564 | 878 | 1197 |
| bnl2 | 1874 | 3134 | 6381 | scfxm3 | 846 | 1317 | 1886 |
| boeing1 | 298 | 373 | 619 | scorpion | 292 | 331 | 411 |
| boeing2 | 125 | 143 | 168 | scrs8 | 447 | 1131 | 783 |
| bore3d | 138 | 188 | 227 | scsd1 | 77 | 760 | 172 |
| brandy | 123 | 205 | 585 | scsd6 | 147 | 1350 | 494 |
| czprob | 689 | 2770 | 2635 | scsd8 | 397 | 2750 | 1548 |
| d6cube | 403 | 6183 | 5883 | sctap1 | 284 | 480 | 643 |
| degen2 | 444 | 534 | 1421 | sctap2 | 1033 | 1880 | 1037 |
| degen3 | 1503 | 1818 | 6398 | sctap3 | 1408 | 2480 | 1339 |
| e226 | 162 | 260 | 598 | seba | 449 | 896 | 766 |

# Data Continued

| Name | $m$ | $n$ | iters | Name | $m$ | $n$ | iters |
|---|---|---|---|---|---|---|---|
| etamacro | 334 | 542 | 1580 | share1b | 107 | 217 | 404 |
| fffff800 | 476 | 817 | 1029 | share2b | 93 | 79 | 189 |
| finnis | 398 | 541 | 680 | shell | 487 | 1476 | 1155 |
| fit1d | 24 | 1026 | 925 | ship04l | 317 | 1915 | 597 |
| fit1p | 627 | 1677 | 15284 | ship04s | 241 | 1291 | 560 |
| forplan | 133 | 415 | 576 | ship08l | 520 | 3149 | 1091 |
| ganges | 1121 | 1493 | 2716 | ship08s | 326 | 1632 | 897 |
| greenbea | 1948 | 4131 | 21476 | ship12l | 687 | 4224 | 1654 |
| grow15 | 300 | 645 | 681 | ship12s | 417 | 1996 | 1360 |
| grow22 | 440 | 946 | 999 | sierra | 1212 | 2016 | 793 |
| grow7 | 140 | 301 | 322 | standata | 301 | 1038 | 74 |
| israel | 163 | 142 | 209 | standmps | 409 | 1038 | 295 |
| kb2 | 43 | 41 | 63 | stocfor1 | 98 | 100 | 81 |
| lotfi | 134 | 300 | 242 | stocfor2 | 2129 | 2015 | 2127 |
| maros | 680 | 1062 | 2998 | | | | |

# A Regression Model for Algorithm Efficiency

*Observed Data:*

$$
\begin{aligned}
t &= \#\text{ of iterations} \\
m &= \#\text{ of constraints} \\
n &= \#\text{ of variables}
\end{aligned}
$$

*Model:*

$$t \approx 2^{\alpha}(m+n)^{\beta}$$

*Linearization:* Take logs:

$$\log t = \alpha \log 2 + \beta \log(m+n) + \underset{\substack{\uparrow \\ \text{error}}}{\epsilon}$$

# Parametric Self-Dual Simplex Method

Recall the thought experiment:

- $\mu$ starts at $\infty$.
- In reducing $\mu$, there are $n+m$ barriers.
- At each iteration, one barrier is passed—the others move about randomly.
- To get $\mu$ to zero, we must on average pass half the barriers.
- Therefore, *on average the algorithm should take $(m+n)/2$ iterations.*

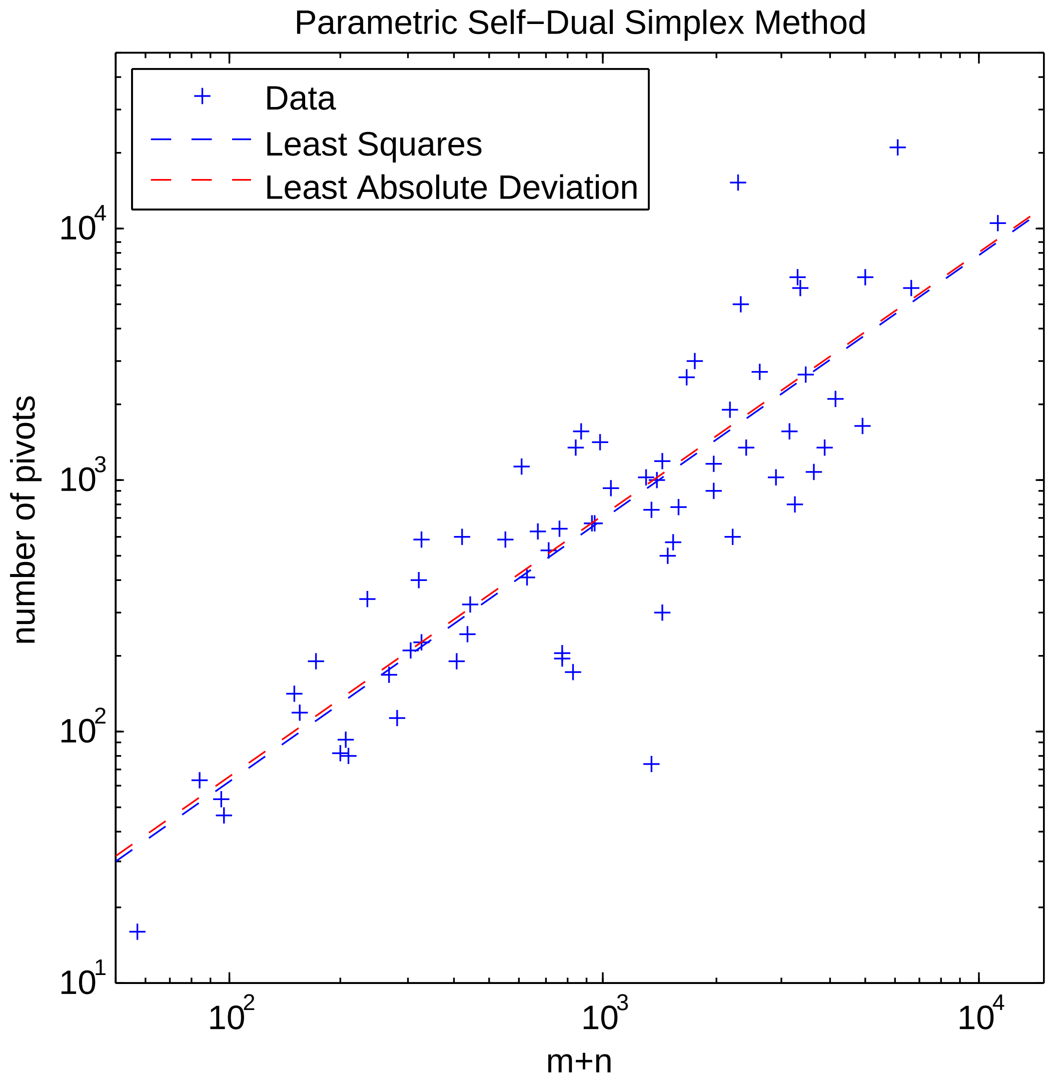
**Using 69 real-world problems from the *Netlib* suite...**

Least Squares Regression:

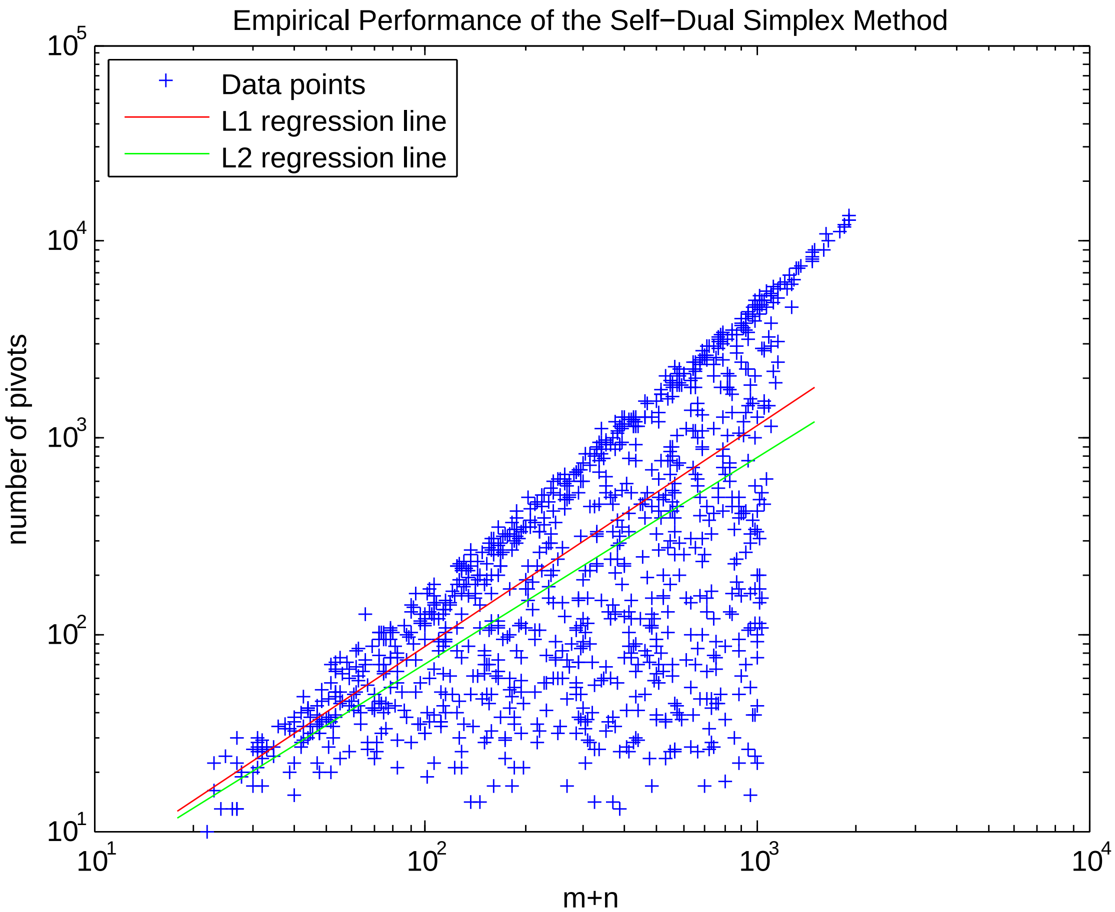
$$\begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix} = \begin{bmatrix} -1.03561 \\ 1.05152 \end{bmatrix} \quad \Longrightarrow \quad T \approx 0.488(m+n)^{1.052}$$

Least Absolute Deviation Regression:

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} -0.9508 \\ 1.0491 \end{bmatrix} \quad \Longrightarrow \quad T \approx 0.517(m+n)^{1.049}$$

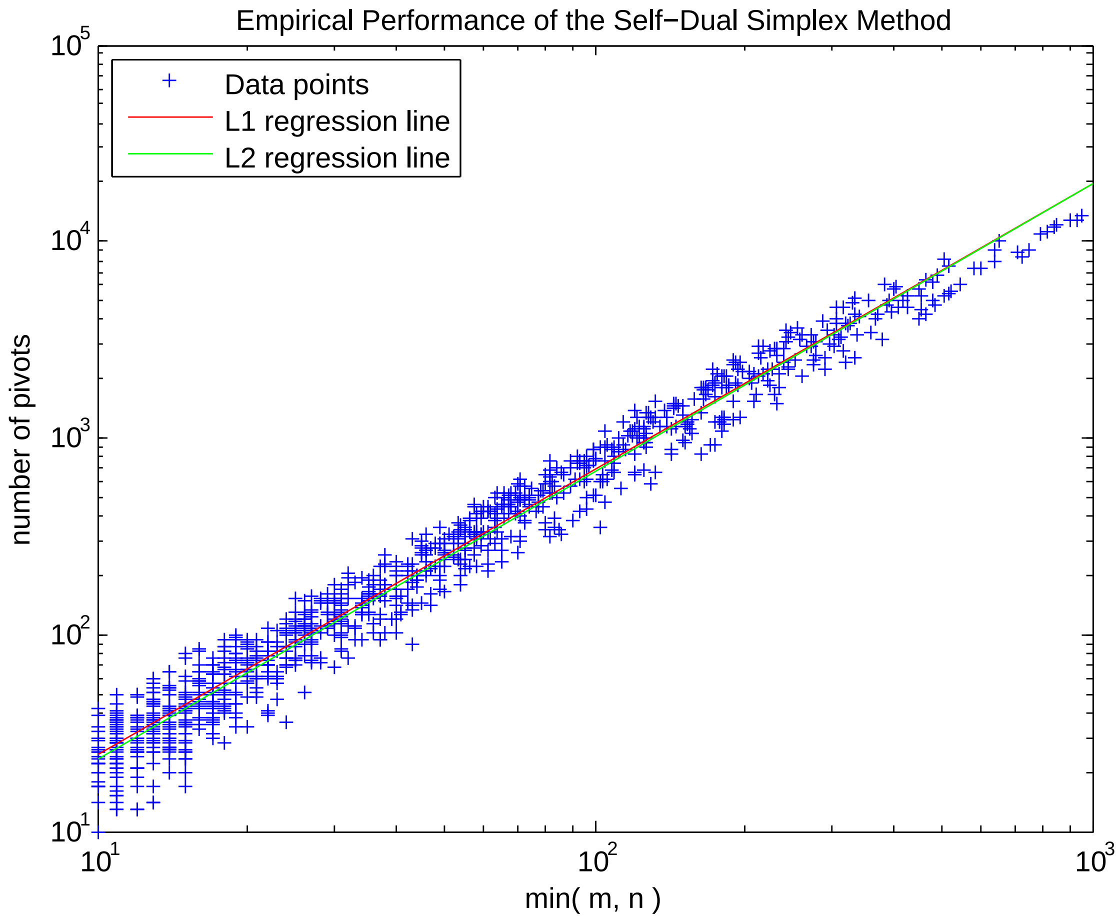



A log–log plot of $T$ vs. $m + n$ and the $L^1$ and $L^2$ regression lines.




$$\text{iters} = 0.486(\text{m} + \text{n})^{1.12}$$

https://vanderbei.princeton.edu/307/python/psd_simplex_pivot.ipynb




$$\text{iters} = 0.8 \min(\text{m}, \text{n})^{1.46}$$

https://vanderbei.princeton.edu/307/python/psd_simplex_pivot.ipynb

## References


[1] I. Adler and N. Megiddo. A simplex algorithm whose average number of steps is bounded between two quadratic functions of the smaller dimension. *Journal of the ACM*, 32:871–895, 1985.

[2] J.R. Birge, R.M. Freund, and R.J. Vanderbei. Prior reduced fill-in in solving equations in interior point algorithms. *OR Letters*, 11:195–198, 1992.

[3] K.-H. Borgwardt. The average number of pivot steps required by the simplex-method is polynomial. *Zeitschrift für Operations Research*, 26:157–177, 1982.

[4] G.B. Dantzig. *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ, 1963.

[5] S.I. Gass and T. Saaty. The computational algorithm for the parametric objective function. *Naval Research Logistics Quarterly*, 2:39–45, 1955.

[6] C.E. Lemke. Bimatrix equilibrium points and mathematical programming. *Management Science*, 11:681–689, 1965.

[7] I.J. Lustig. The equivalence of Dantzig's self-dual parametric algorithm for linear programs to Lemke's algorithm. Technical Report SOL 87-4, Department of Operations Research, Stanford University, 1987.

[8] J.L. Nazareth. Homotopy techniques in linear programming. *Algorithmica*, 1:529–535, 1986.

[9] B. Rudloff, F. Ulus, and R.J. Vanderbei. A parametric simplex algorithm for linear vector optimization problems. *Mathematical Programming, Series A*, 163(1):213–242, 2017.

[10] S. Smale. On the average number of steps of the simplex method of linear programming. *Mathematical Programming*, 27:241–262, 1983.

[11] R.J. Vanderbei. *Linear Programming: Foundations and Extensions*. Springer, 4th edition, 2013.

# Thank You!

# Questions?

# Some Acknowledgements

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