Extinguishing Poisson’s Spot
with Linear Programming

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http://www.princeton.edu/~rvdb
Are We Alone?
Indirect Detection Methods

Over 300 planets found—more all the time
Wobble Methods

Radial Velocity (RV).
For approximately edge-on systems.
Measure periodic doppler shift.

Astrometry.
Best for face-on systems.
Measure circular wobble against background stars.
Transit Method

- HD209458b confirmed both via RV and transit.
- Period: 3.5 days
- Separation: 0.045 AU (0.001 arcsecs)
- Radius: $1.3R_J$
- Intensity Dip: $\sim 1.7\%$
- Venus Dip = 0.01%, Jupiter Dip: 1%

Venus Transit (R.J. Vanderbei)

Credit: Hans Deeg, from 'Transits of Extrasolar Planets'
Direct Imaging

Fomalhaut.

*Distance:* 25 light-years  \( \textbf{Star-Planet Separation:} \) 100 AU
Terrestrial Planet Finder Telescope (TPF)

- **DETECT**: Search 150-500 nearby (5-15 pc distant) Sun-like stars for Earth-like planets.
- **CHARACTERIZE**: Determine basic physical properties and measure “biomarkers”, indicators of life or conditions suitable to support it.

**Why Is It Hard? Can’t Hubble do it?**

- If the star is Sun-like and the planet is Earth-like, then the reflected visible light from the planet is $10^{-10}$ times as bright as the star.
- If the star is 10 pc (33 ly) away and the planet is 1 AU from the star, the angular separation is at most 0.1 arcseconds!
- A point source (i.e. star) produces not a point image but an *Airy pattern* consisting of an *Airy disk* surrounded by a system of *diffraction rings* completely swamping light from the nearby planet.
- By *apodizing* the entrance pupil, one can control the shape and strength of the diffraction rings.
Space Occulter Design for Planet-Finding
Space-based Occulter (TPF-O)

Telescope Aperture: 4m, Occulter Diameter: 50m, Occulter Distance: 72,000km
Plain External Occulter (Doesn’t Work!)

Shadow ⇒

Circular Occulter ⇓

Note bright spot at center (Poisson’s spot)

Telescope Image

Shadow (Log Stretch)
Shaped Occulter—Eliminates Poisson’s Spot

Shadow $\Rightarrow$

Shaped Occulter

$\Downarrow$

$\leftarrow$ Bright spot is gone

$\uparrow$

Telescope image shows planet

$\leftarrow$ Shadow is dark (Log Stretch)
Apodized Occulters

- The problem (as mentioned before) is \textit{diffraction}.
- Abrupt edges create unwanted diffraction.
- \textbf{Solution:} Soften the edges with a partially transmitting material—an \textit{apodizer}.
- Let $A(r, \theta)$ denote \textit{attenuation} at location $(r, \theta)$ on the occulter.
- The \textit{intensity} of the downstream light is given by the square of the magnitude of the electric field $E(\rho, \phi)$.
- \textit{Babinet’s principle} plus \textit{Fresnel propagation} gives a formula for the downstream electric field:

$$E(\rho, \phi) = 1 - \frac{1}{i\lambda z} \int_0^\infty \int_0^{2\pi} e^{\frac{ir}{\lambda z}(r^2 + \rho^2 - 2r\rho \cos(\theta - \phi))} A(r, \theta) r d\theta dr.$$  

where

- $z$ is distance “downstream” and
- $\lambda$ is wavelength of light.
Attenuation Profile Optimization

minimize $\gamma$

subject to

$-\gamma \leq \Re(E(\rho)) \leq \gamma$ for $\rho \in \mathcal{R}$, $\lambda \in \Lambda$

$-\gamma \leq \Im(E(\rho)) \leq \gamma$ for $\rho \in \mathcal{R}$, $\lambda \in \Lambda$

$A'(r) \leq 0$ for $0 \leq r \leq R$

$-d \leq A''(r) \leq d$ for $0 \leq r \leq R$

Specific choice:

$R = 25$, $d = 0.04$, $\mathcal{R} = [0, 3]$, $\Lambda = [0.4, 1.1] \times 10^{-6}$

where all metric quantities are in meters.

An infinite dimensional linear programming problem.

Discretize:

- $[0, R]$ into 5000 evenly space points.
- $\mathcal{R}$ into 150 evenly spaced points.
- $\Lambda$ into increments of $0.1 \times 10^{-6}$. 


Petal-Shaped Occulters

- From Jacobi-Anger expansion we get:

\[
E(\rho, \phi) = 1 - \frac{2\pi}{i \lambda z} \int_0^R e^{\frac{i\pi}{\lambda z}(r^2 + \rho^2)} J_0\left(\frac{2\pi r \rho}{\lambda z}\right) A(r) r dr
\]

\[
- \sum_{k=1}^{\infty} \frac{2\pi (-1)^k}{i \lambda z} \left( \int_0^R e^{\frac{i\pi}{\lambda z}(r^2 + \rho^2)} J_k N \left(\frac{2\pi r \rho}{\lambda z}\right) \frac{\sin(\pi k A(r))}{\pi k} r dr \right)
\]

\[
\times \left(2 \cos(k N (\phi - \frac{\pi}{2}))\right)
\]

where \( N \) is the number of petals.

- For small \( \rho \), truncated summation well-approximates full sum.

- Truncated after 10 terms.

- \( \lambda \in [0.4, 1.1] \) microns.

- \( z = 72,000 \) km, \( R = 25 \) m.

- In angular terms, \( R/z = 0.073 \) arcseconds.
Our Solar System As Seen From Fomalhaut
Optimal vs. Sub-Optimal But Explicit Design

Size (tip-to-tip):
- Optimal: 50 meters
- Hypergaussian: 104 meters

Distance:
- Optimal: 72,000 km
- Hypergaussian: 150,000 km