

# Linear, Conic, and Non-Convex Formulations of the High Contrast Imaging Problem

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# High-Contrast Imaging

Objective:

$$\underset{A(x,y):(x,y) \in \mathcal{P}}{\text{maximize}} \iint_{(x,y) \in \mathcal{P}} A(x,y) dx dy$$

Constraints:

$$\left\| \iint_{(x,y) \in \mathcal{P}} e^{2\pi x \xi i} A(x,y) e^{2\pi y \eta i} dx dy \right\| \leq 10^{-5} \iint_{(x,y) \in \mathcal{P}} A(x,y) dx dy, \quad (\xi, \eta) \in \mathcal{D}$$
$$0 \leq A(x,y) \leq 1, \quad (x,y) \in \mathcal{P}$$

where

$$\mathcal{P} = \{(x,y) \mid x^2 + y^2 \leq (1/2)^2\}$$

and

$$\mathcal{D} = \{(\xi, \eta) \mid \rho_0^2 \leq \xi^2 + \eta^2 \leq \rho_1^2 \text{ and } \eta \leq \xi\}$$

and  $\rho_0$  and  $\rho_1$  are given *inner* and *outer working angles* and  $i = \sqrt{-1}$ .

Note: The norm is the norm of a complex number—it's the sum of the squares of the real and imaginary parts.

# A Big Problem

The sets  $\mathcal{P}$  and  $\mathcal{D}$  each contain a continuum of points.

Hence, the problem has an uncountably infinite number of variables and an uncountably infinite number of constraints.

Our aim is to solve the problem using discrete samplings of  $\mathcal{P}$  and  $\mathcal{D}$ .

Such discretizations lead to a *finite-dimensional* convex (in fact *conic*) optimization problem.

The finer the discretization, the better.

# Fourier Formulation

Objective:

$$\begin{array}{l} \text{maximize} \\ A(x,y) : (x,y) \in \mathcal{P} \\ E(\xi,\eta) : (\xi,\eta) \in \mathcal{D} \end{array} \iint_{(x,y) \in \mathcal{P}} A(x,y) dx dy$$

Constraints:

$$\begin{aligned} E(\xi, \eta) &= \iint_{(x,y) \in \mathcal{P}} e^{2\pi x \xi i} A(x, y) e^{2\pi y \eta i} dx dy, & (\xi, \eta) \in \mathcal{D} \\ \|E(\xi, \eta)\| &\leq 10^{-5} E(0, 0), & (\xi, \eta) \in \mathcal{D} \\ 0 &\leq A(x, y) \leq 1, & (x, y) \in \mathcal{P} \end{aligned}$$

Note:  $A(\cdot, \cdot)$  is a real-valued function but  $E(\cdot, \cdot)$  is complex valued.

# Fast Fourier Formulation

Objective:

$$\begin{array}{l} \text{maximize} \\ A(x,y) : (x,y) \in \mathcal{P} \\ C(\xi,y) : \xi \in [-\rho_1, \rho_1], y \in [-1,1] \\ E(\xi,\eta) : (\xi,\eta) \in \mathcal{D} \end{array} \iint_{(x,y) \in \mathcal{P}} A(x,y) dx dy$$

Constraints:

$$C(\xi, y) = \int_{x : (x,y) \in \mathcal{P}} e^{2\pi x \xi i} A(x, y) dx, \quad \begin{array}{l} -\rho_1 \leq \xi \leq \rho_1 \\ -1 \leq y \leq 1 \end{array}$$

$$E(\xi, \eta) = \int_{-1}^1 C(\xi, y) e^{2\pi y \eta i} dy, \quad (\xi, \eta) \in \mathcal{D}$$

$$\|E(\xi, \eta)\| \leq 10^{-5} E(0, 0), \quad (\xi, \eta) \in \mathcal{D}$$

$$0 \leq A(x, y) \leq 1, \quad (x, y) \in \mathcal{P}$$

# Symmetrizing

The original problem, as formulated, is completely symmetric with respect to flipping about the  $x$  and  $y$  axes.

The optimal solution will have the same symmetry:

$$A(x, y) = A(-x, y) = A(x, -y) = A(-x, -y)$$

Hence, the problem can be expressed using variables defined over just the first quadrant.

The symmetry also implies that  $C(\cdot, \cdot)$  and  $E(\cdot, \cdot)$  are also *symmetric* and purely *real* and therefore the *norm* is just an *absolute value*.

Hence, the *cone constraints* can be reduced to simple inequalities and the problem becomes a *linear programming* problem...

# Symmetric Fast Fourier Formulation

Objective:

$$\begin{array}{l} \text{maximize} \\ A(x,y) : (x,y) \in \mathcal{P}_+ \\ C(\xi,y) : \xi \in [-\rho_1, \rho_1], y \in [-1,1] \\ E(\xi,\eta) : (\xi,\eta) \in \mathcal{D}_+ \end{array} \iint_{(x,y) \in \mathcal{P}_+} A(x,y) dx dy$$

Constraints:

$$C(\xi, y) = \int_{x : (x,y) \in \mathcal{P}_+} \cos(2\pi x \xi) A(x, y) dx, \quad \begin{array}{l} -\rho_1 \leq \xi \leq \rho_1 \\ -1 \leq y \leq 1 \end{array}$$

$$E(\xi, \eta) = \int_{-1}^1 C(\xi, y) \cos(2\pi y \eta) dy, \quad (\xi, \eta) \in \mathcal{D}_+$$

$$-10^{-5} E(0, 0) \leq E(\xi, \eta) \leq 10^{-5} E(0, 0), \quad (\xi, \eta) \in \mathcal{D}_+$$

$$0 \leq A(x, y) \leq 1, \quad (x, y) \in \mathcal{P}$$

# Problem Stats

Discretize the square  $[-1, 1]^2$  into a discrete set of  $100 \times 100$  and define  $\mathcal{P}$  as the appropriate subset of this grid.

Discretize the square  $[-\rho_1, \rho_1]^2$  into a discrete set of  $71 \times 71$  and define  $\mathcal{D}$  as the appropriate subset of this grid.

Formulation	variables	constraints	nonz( $A$ )	nonz( $Q$ )	nonz( $L$ )	Arith. Ops.
Original Formulation	7,825	1,637	12,809,525	61,230,625	43,420,925	260,044,071,562
Fourier	17,906	11,718	78,880,057	6,548	188,361,433	1,394,426,403,936
Fast Fourier	31,865	25,677	3,083,695	6,548	71,947,373	398,410,536,128
Symmetric Fast Fourier	5,061	3,944	140,332	0	1,294,385	863,956,159

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Symmetric Fast Fourier	5,061	3,944	140,332	0	1,294,385	863,956,159
Symmetric Fourier	3,261	2,144	2,549,632	0	7,026,326	12,563,316,385
Symmetric Orig. Form.	1,965	848	1,666,320	0	2,025,448	1,622,025,837

# Moral

Mathematically equivalent formulations of a problem can be dramatically easier or harder to solve.

It's hard to predict.

*"The simplex method is 200 times faster than the simplex method." – John Forrest*

Thank You!

And Let The Fun Begin...

