Stability of Ring Systems

Robert J. Vanderbei

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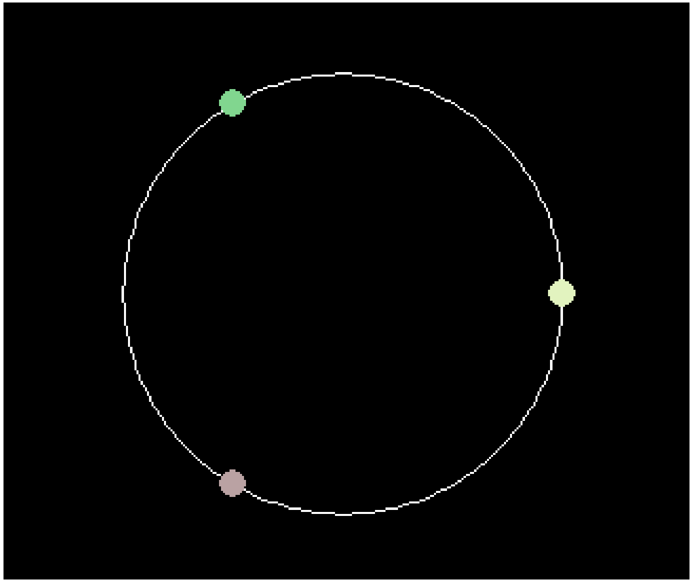
PACM
Princeton University

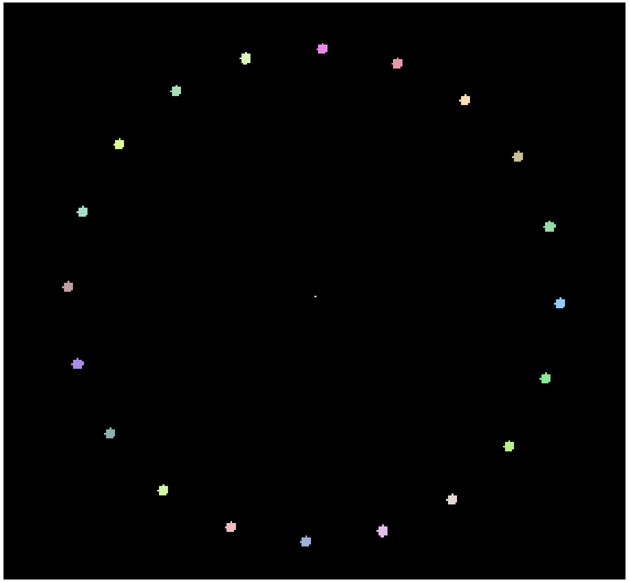
Linear Stability of Ring Systems, *Astronomical Journal*, *133*:656-664, 2007

Linear Stability of Ring Systems Around Oblate Central Masses, *Advances in Space Research*, *42*:1370–1377, 2008

http://www.princeton.edu/∼rvdb

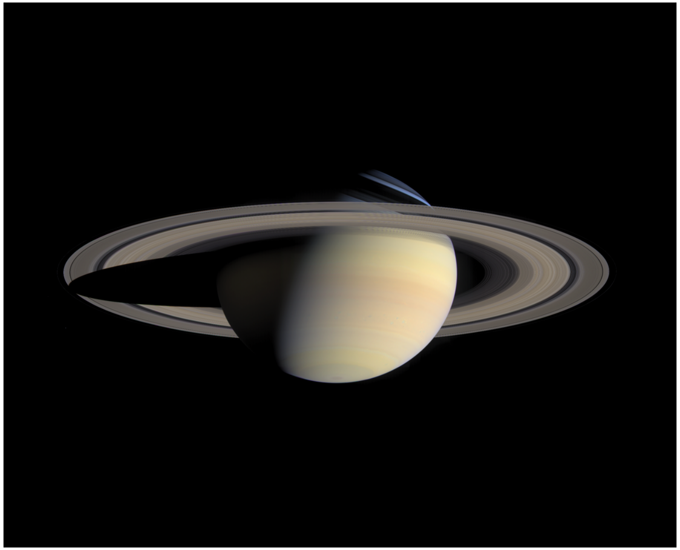
Isolated Ring Systems Are Unstable



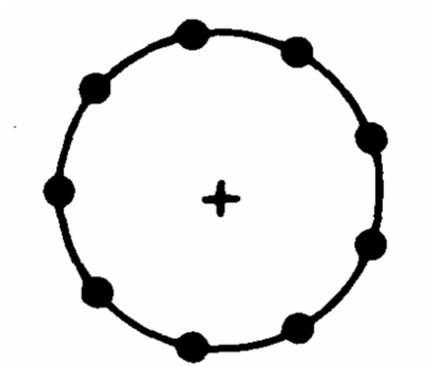


Theorem 1 *The system is stable if and only if* $n = 2$.

Saturn's Rings



Beautiful Saturn



Simplified model of a ring system

In 1859, J.C. Maxwell won the prestigious Adams Prize.

His Results:

- Rings of Saturn must be composed of small particles.
- Modeled the ring as n co-orbital particles of mass m.
- For large n, ring system is stable if

$$\frac{m}{M} \leq \frac{2.298}{n^3}$$

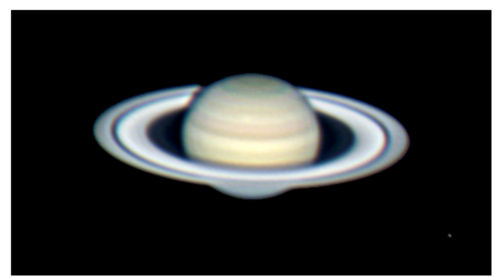
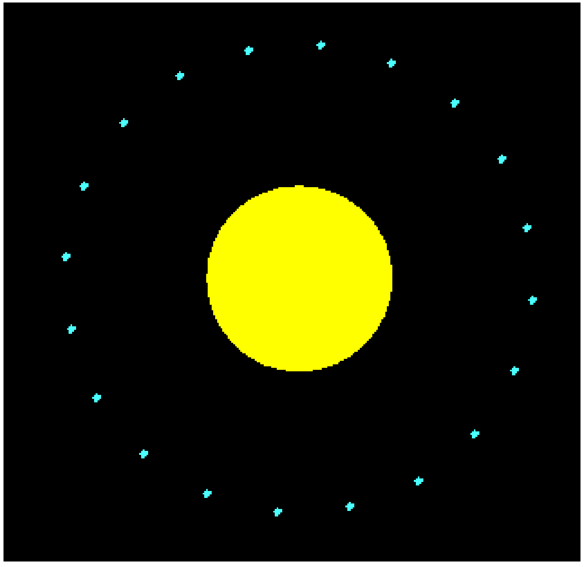


Image From Earth

A Large Central Mass Stabilizes



Saturn and 20 Janus-mass moons

Stable! WHY?

Common misconception: the massive body dominates the dynamics dwarfing the moon-moon interactions.

This is WRONG.

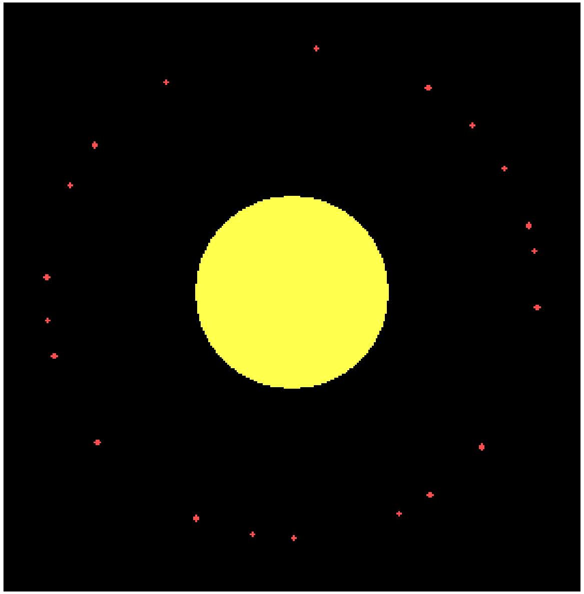
Slight Perturbation

Here, again, are 20 Janus masses

Orbits are initialized to be circular

Distances from Saturn are randomized (only slightly)

Note the effective repulsion!



Main Result

R. J. Vanderbei and E. Kolemen *Linear Stability of Ring Systems*. *Astronomical Journal*, 133:656-664, 2007.

Theorem 2

- *For* $2 \leq n \leq 6$*, the ring system is unstable.*
- *For* $n \geq 7$*, the ring system is (linearly) stable if and only if*

$$\frac{m}{M} \leq \frac{\gamma_n}{n^3}.$$

- $\lim_{n\to\infty} \gamma_n = 2.2987$.

Simulation confirms the stability analysis:

n	γ_n	Simulator
2	*	[0.0, 0.007]
6	*	[0.0, 0.025]
7	2.452	[2.45, 2.46]
8	2.4121	[2.41, 2.42]
10	2.3753	[2.37, 2.38]
12	2.3543	[2.35, 2.36]
14	2.3411	[2.34, 2.35]
20	2.3213	[2.32, 2.33]
36	2.3066	[2.30, 2.31]
50	2.3031	[2.30, 2.31]
100	2.2999	[2.30, 2.31]
500	2.2987	

The Formula For γ_n Is Explicit But Ugly

$$
\begin{aligned}
n^3/\gamma_n \;=\;& 2(J_n - \tilde{J}_{n/2\pm 1,n}) + \frac{9}{2}(J_n - \tilde{J}_{n/2,n}) - 5I_n \\
&+\sqrt{\left(2(J_n - \tilde{J}_{n/2\pm 1,n}) + \frac{9}{2}(J_n - \tilde{J}_{n/2,n}) - 4I_n\right)^2 - \frac{9}{4}\left(J_n - \tilde{J}_{n/2,n}\right)^2},
\end{aligned}
$$

where

$$
\begin{aligned}
I_n &= \frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{\sin\left(\pi k/n\right)} \\
J_n &= \frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{\sin^3\left(\pi k/n\right)} \\
\tilde{J}_{j,n} &= \frac{1}{4}\sum_{k=1}^{n-1}\frac{\cos(2\pi kj/n)}{\sin^3(\pi k/n)}
\end{aligned}
$$

Asymptotics

For n large,

$$I_n \;\approx\; \frac{n}{2\pi} \sum_{k=1}^{(n-1)/2} \frac{1}{k} \;\approx\; \frac{n}{2\pi}\, \log(n/2)$$

$$J_n \;\approx\; \frac{n^3}{2\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \;=\; \frac{n^3}{2\pi^3}\, \zeta(3) = 0.01938\, n^3$$

$$\tilde{J}_{n/2,n} \;\approx\; -\frac{3}{4} J_n.$$

Hence,

$$\gamma_n \approx \frac{1}{\frac{7}{8}(13+\sqrt{160})J_n/n^3} \approx 2.2987.$$

Oblateness

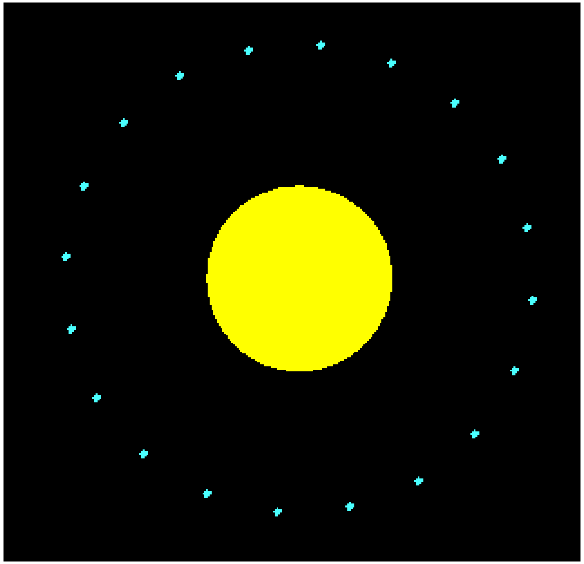
If the central body is oblate with oblateness parameter $\mathcal{J}_2$ and equatorial radius R, a similar analysis yields, for large n,

$$\gamma_n \approx \frac{8}{7} \frac{(1 - \frac{3}{2}\mathcal{J}_2 \left(\frac{R}{r}\right)^2)^2}{13 - \frac{57}{2}\mathcal{J}_2 \left(\frac{R}{r}\right)^2 + \sqrt{(13 - \frac{57}{2}\mathcal{J}_2 \left(\frac{R}{r}\right)^2)^2 - 9(1 - \frac{3}{2}\mathcal{J}_2 \left(\frac{R}{r}\right)^2)^2}} \frac{n^3}{J_n}$$

For Saturn, $\mathcal{J}_2 = 1.6297 \times 10^{-2}$ and $R/r = 0.3967$. With these values, we get

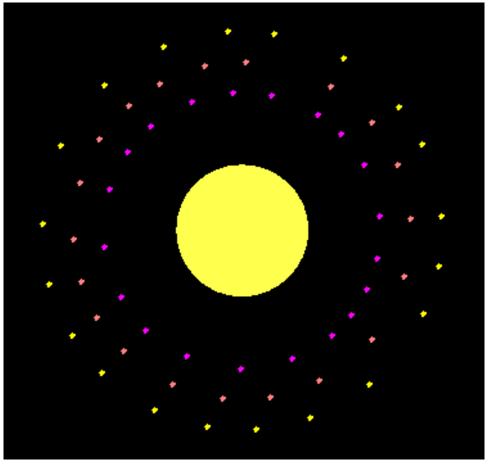
$$\gamma_n \approx 2.2945.$$

From simulator with $n = 60$, 2.280 is stable whereas 2.281 is not.



Rings at Multiple Radii

General principle: it is easier for a body to destabilize bodies at the same radius from the central mass.



Hence, if each of many single rings are stable, then one might expect the entire system to be stable.

Mathematical verification is profoundly difficult—no longer does a single counter-rotation freeze all bodies.

Linear Density Estimate

Let

$$\lambda \;=\; \text{linear density of the masses} = \frac{\text{diam of a boulder}}{\text{separation between boulders}}$$

The density of the boulders in Saturn's rings is about $1/8$ of Earth's density.

Assuming a central mass equal to Saturn and a ring radius of $120,000$km, we get

$$\lambda \leq 20.4\%.$$

Remark: Gravity scales correctly—a marble orbits a bowling ball every 90 minutes.

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SATURN MOVIE