Two Topics: A New Flat Map and Imaging Exoplanets

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In 1569, when Flemish cartographer Geradus Mercator flattened the earth’s cylindrical surface onto paper, he gave sailors the tools to navigate ocean voyages. But he also distorted the size of countries nearest the poles—North America appears abnormally large, for example. Despite the inaccuracies, the Mercator projection became the norm, and was even the basis of Google Maps until as recently as 2018. Astrophysicist J. Richard Gott, along with colleagues David Goldberg and Robert Vanderbei, set out in 2019 to fix the inaccuracies and came up with a double-sided map that is similar to a vinyl record in shape. It improves geographical parity between continents, finally representing the southern hemisphere as fairly as possible on paper. The map is free to access online, and the scientists are working with publishers to make it widely available for sale in the future. —Eloise Barry
Classic Map Projections...
Winkel Tripel Map
Azimuthal Equidistant
Distortion Metrics

- Skewness
- Flexion
- Isotropy
- Area
- Distances
- Boundary Cuts

Winkel Tripel is “best”
A New Map Projection...
It’s a Two-Sided Map
Let’s Minimize Stress
Imagine a Large Rubber Earth Ball
Suppose it has an Expandable Metal Ring Inside
Suppose that the $y$-axis is the polar axis and hence the equatorial ring is in the $(x, z)$ plane.

Now let’s **stretch** the ring so that it has a radius larger than its default value of, say, 1.

Without loss of generality, we can focus on just one longitudinal plane, let’s say the one associated with $z = 0$.

As shown above, the geometry of the stretched ball can be described by two functions $f$ and $g$. 
Let $x$ and $y$ denote the coordinates of the *unstretched* ball and let $\tilde{x}$ and $\tilde{y}$ denote the coordinates of the *stretched* ball.

If we let $\theta$ denote the angle down from the North Pole, then we have

$$x(\theta) = \sin(\theta), \; y(\theta) = \cos(\theta) \quad \text{and} \quad \tilde{x}(\theta) = f(\theta), \; \tilde{y}(\theta) = g(\theta)$$

According to physics, the shape of the stretched ball will be such that the integral over the ball’s surface of the magnitude squared of the stress tensor is *minimized*. 
Stress

At the point \((\tilde{x}(\theta), \tilde{y}(\theta))\) in the stretched circular slice, let \(\sigma(\theta)\) denote the stress in the direction tangent to the circle and let \(\rho(\theta)\) denote the stress in the direction perpendicular to the 2-dimensional plane of the slice.

Working with infinitesimal perturbations, we have

\[
\|(dx, dy)\| = \sqrt{dx^2 + dy^2} = \sqrt{\cos^2(\theta) + \sin^2(\theta)} \, d\theta = d\theta.
\]

and

\[
\|(d\tilde{x}, d\tilde{y})\| = \sqrt{d\tilde{x}^2 + d\tilde{y}^2} = \sqrt{(f'(\theta))^2 + (g'(\theta))^2} \, d\theta.
\]

and from these it is easy to compute \(\sigma(\theta)\):

\[
\sigma(\theta) = \frac{\|(d\tilde{x}, d\tilde{y})\|}{\|(dx, dy)\|} - 1 = \sqrt{(f'(\theta))^2 + (g'(\theta))^2} - 1.
\]

Computing \(\rho(\theta)\) is even easier:

\[
\rho(\theta) = \frac{f(\theta)}{\sin(\theta)} - 1.
\]
Minimum Stress Problem

\[
\min_{f, g} \int_0^{\pi/2} \left( \sigma(\theta)^2 + \rho(\theta)^2 \right) 2\pi \sin(\theta) d\theta
\]

where

\[
\sigma(\theta) = \sqrt{f'(\theta)^2 + g'(\theta)^2} - 1
\]

\[
\rho(\theta) = \frac{f(\theta)}{\sin(\theta)} - 1
\]

\[
g(\theta) = 0, \quad 0 \leq \theta \leq \pi/2
\]

\[
f(0) = 0
\]

\[
f'(\theta) \geq 0, \quad 0 \leq \theta \leq \pi/2.
\]
Question:

Is the optimal function linear:  \( f(\theta) = c\theta \)?
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Conjecture:
Maybe
Objective Function:

\[ S(f) = \int_0^{\pi/2} \left( (f'(\theta) - 1)^2 + \left( \frac{f(\theta)}{\sin(\theta)} - 1 \right)^2 \right) 2\pi \sin(\theta) d\theta \]

Perturbation:

\[ \partial f(\theta), \quad 0 \leq \theta \leq \pi/2 \]

Critical Points:

\[ \lim_{\varepsilon \to 0} \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon} = 0 \]
Compute the Ratio:

\[
\frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon} = 2\pi \int_0^{\pi/2} \left( f'(\theta)^2 + 2f'(\theta)\varepsilon \partial f'(\theta) + \varepsilon^2 \partial f'(\theta)^2 - 2f'(\theta) - 2\varepsilon \partial f'(\theta) + 1 \\
- f'(\theta)^2 + 2f'(\theta) + 1 \\
+ \frac{f(\theta)^2}{\sin^2(\theta)} + 2\frac{f(\theta)\varepsilon \partial f(\theta)}{\sin^2(\theta)} + \varepsilon^2 \frac{\partial f(\theta)^2}{\sin^2(\theta)} - 2\frac{f(\theta)}{\sin(\theta)} - 2\frac{\varepsilon \partial f(\theta)}{\sin(\theta)} + 1 \\
- \frac{f(\theta)^2}{\sin^2(\theta)} + 2\frac{f(\theta)}{\sin(\theta)} - 1 \right) \sin(\theta) \frac{1}{\varepsilon} d\theta.
\]

\[
= 2\pi \int_0^{\pi/2} \begin{pmatrix} 2f'(\theta) \partial f'(\theta) + \varepsilon \partial f'(\theta)^2 \\
+ 2\frac{f(\theta) \partial f(\theta)}{\sin^2(\theta)} + \varepsilon \frac{\partial f(\theta)^2}{\sin^2(\theta)} \\
- 2 \frac{\partial f(\theta)}{\sin(\theta)} \end{pmatrix} \sin(\theta) d\theta.
\]
Take the Limit:

\[
\frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon} = 2\pi \int_{0}^{\pi/2} \left( 2f'(\theta) \partial f'(\theta) + \varepsilon \partial f'(\theta)^2 - 2\partial f'(\theta) \\
+ 2f(\theta) \partial f(\theta) + \varepsilon \frac{\partial f(\theta)^2}{\sin^2(\theta)} - 2\frac{\partial f(\theta)}{\sin(\theta)} \right) \sin(\theta) d\theta
\]

\[
\lim_{\varepsilon \to 0} \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon} = 4\pi \int_{0}^{\pi/2} \left( f'(\theta) \partial f'(\theta) - \partial f'(\theta) \\
+ \frac{f(\theta) \partial f(\theta)}{\sin^2(\theta)} - \frac{\partial f(\theta)}{\sin(\theta)} \right) \sin(\theta) d\theta.
\]

Simpler notation...

\[
\frac{\partial S}{\partial f} = \lim_{\varepsilon \to 0} \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon}
\]
Critical Point:

Set the differential to zero...

\[ \frac{\partial S}{\partial f} = 4\pi \int_{0}^{\pi/2} \left( f'(\theta) \partial f'(\theta) - \partial f'(\theta) + \frac{f(\theta)}{\sin^2(\theta)} \partial f(\theta) - \frac{\partial f(\theta)}{\sin(\theta)} \right) \sin(\theta) \, d\theta \]

\[ = 4\pi \int_{0}^{\pi/2} \left( (f'(\theta) - 1) \partial f'(\theta) + \left( \frac{f(\theta)}{\sin(\theta)} - 1 \right) \partial f(\theta) \right) \sin(\theta) \, d\theta \]

\[ = 0. \]
Integrate by Parts:

\[
\int_0^{\pi/2} (f'(\theta) - 1) \sin(\theta) \, \partial f'(\theta) \, d\theta = (f'(\pi/2) - 1) \, \partial f(\pi/2) - \int_0^{\pi/2} \left( f''(\theta) \sin(\theta) + (f'(\theta) - 1) \cos(\theta) \right) \, \partial f(\theta) \, d\theta.
\]

Substituting this into our equation defining critical points, we get

\[
0 = (f'(\pi/2) - 1) \, \partial f(\pi/2) - \int_0^{\pi/2} \left( f''(\theta) \sin(\theta) + (f'(\theta) - 1) \cos(\theta) - \frac{f(\theta)}{\sin(\theta)} + 1 \right) \, \partial f(\theta) \, d\theta.
\]

This equation must be equal to zero for all valid choices of the perturbation function \( \partial f \). Hence...
Differential Equation:

\[
\sin^2(\theta) f''(\theta) + \sin(\theta) \cos(\theta) f'(\theta) - f(\theta) = \sin(\theta) \cos(\theta) - \sin(\theta)
\]

\[
f(0) = 0
\]

\[
f'(\pi/2) = 1.
\]
Differential Equation:

\[
\sin^2(\theta) f''(\theta) + \sin(\theta) \cos(\theta) f'(\theta) - f(\theta) = \sin(\theta) \cos(\theta) - \sin(\theta)
\]

\[f(0) = 0\]

\[f'(\pi/2) = 1.\]

Let’s try \( f(\theta) = \theta \ldots \)

\[
\sin^2(\theta) f''(\theta) + \sin(\theta) \cos(\theta) f'(\theta) - f(\theta) = \sin(\theta) \cos(\theta) - \theta
\]

\[\neq \sin(\theta) \cos(\theta) - \sin(\theta)\]

Almost but no cigar. 😐
Mathematica

\[
\begin{align*}
  s &= \text{DSolve}\left\{ \{\sin[x]^2 y''[x] + \sin[x] \cos[x] y'[x] - y[x] == \sin[x] \cos[x] - \sin[x], \right. \\
  &\quad \left. y[0] == 0, y'[\pi/2] == 1 \}\}, y[x], x \right\} \text{ // FullSimplify} \\
  f[x_] &= y[x] / . s[[1]]
\end{align*}
\]

The output produced by \textit{Mathematica} (with \(x\) changed to \(\theta\)) is

\[
f(\theta) = \log(2) \tan(\theta/2) - 2 \cot(\theta/2) \log(\cos(\theta/2)).
\]

Matlab

```
syms f(x)
f1 = diff(f,x);
f2 = diff(f,x,2);
ode = \sin(x)^2 * f2 + \sin(x)\cos(x) * f1 - f == \sin(x)\cos(x) - \sin(x);
cond1 = f(0) == 0;
cond2 = f1(pi/2) == 1;
conds = [cond1 cond2];
fSol(x) = dsolve(node,conds)
fSim(x) = simplify(fSol(x), 'steps', 14)
```

The output produced by \textit{Matlab} (again with \(x\) changed to \(\theta\)) is

\[
f(\theta) = -\frac{\log(\cos(\theta)/4 + 1/4) + 2 \log(e^{i\theta} + 1) \cos(\theta) - \log(2) \cos(\theta) - \theta \cos(\theta) i}{\sin(\theta)}
\]

\textbf{NOTE:} These two functions look different, but they are the same.
Check that it’s a Min, not a Max or a Saddle Point

Let’s look at the second order differential in every possible perturbational direction...

$\frac{\partial^2 S}{\partial f^2} = \lim_{\varepsilon \to 0} \frac{S(f + \varepsilon \partial f) - 2S(f) + S(f - \varepsilon \partial f)}{\varepsilon^2}$

Let’s compute...

$\frac{S(f + \varepsilon \partial f) - 2S(f) + S(f - \varepsilon \partial f)}{\varepsilon^2} = \frac{S(f + \varepsilon \partial f) - S(f)}{\varepsilon^2} + \frac{S(f - \varepsilon \partial f) - S(f)}{\varepsilon^2}$

$= 4\pi \int_0^{\pi/2} \left( \partial f'(\theta)^2 + \frac{\partial f(\theta)^2}{\sin^2(\theta)} \right) \sin(\theta) d\theta$

$\geq 0.$

Ergo, it’s a minimum!
Are We Alone? What Are The Odds?

This is Earth
Some Background
Christiaan Huygens (1678): Light is a Wave

Young's two-slit diffraction experiment (1801):
James Clerk Maxwell (1862): Light is an Electro-Magnetic Wave

And God Said

\[ \nabla \cdot \vec{D} = \rho_{\text{free}} \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ \nabla \times \vec{H} = \vec{J}_{\text{free}} + \frac{\partial \vec{D}}{\partial t} \]

and then there was light.

Light wave

\[ \lambda = \text{wave length} \]
\[ E = \text{amplitude of electric field} \]
\[ M = \text{amplitude of magnetic field} \]

distance
Albert Einstein (1905): Light is a Particle

Explained the photoelectric effect, which led to the new field of quantum mechanics. Einstein himself never accepted it.

Modern CCD cameras count photons.
Direct Detection
First Detection via Direct Imaging
Mag. 1.2, Distance 25 ly, Imaged by HST, Period: 872 years,

Fomalhaut
HST ACS/HRC

Dust ring
Location of Fomalhaut
Scattered starlight “noise”
Coronagraph mask

No data
Background Star

100 AU 13“
Why It’s Hard

Premise: If there is intelligent life “out there”, it is probably similar to life as we know it on Earth.

- **Bright Star/Faint Planet:** In visible light, our Sun is ten billion times brighter than Earth. That’s 25 mags.

- **Close to Each Other:** A planet at 1 AU from a star at 10 parsecs (33 lightyears) can appear at most 0.1 arcseconds in separation.

- **Far from Us:** There are less than 100 Sun-like stars within 10 parsecs.
Can Ground-Based Telescopes Do It?

- Atmospheric distortion limits *resolution* to about 1 arcsec.
  Note: Resolution refers to equally bright objects. If one is much brighter than the other, then it is more difficult.

- Segmented mirrors limit contrast

- Current adaptive optics not good enough

No they can’t (at least not yet)!
Can Hubble Do It?

No it can’t!

The problem is diffraction

Would have to be $1000 \times$ bigger (in each dimension!)
Concept 1: Shaped Pupil Coronagraph
Diffraction Control via Shaped Pupils

- Focal plane
- Light cone
- Pupil plane
A key goal in high-contrast imaging is to maximize light through an apodized circular aperture subject to the constraint that virtually no light reaches a given dark zone $\mathcal{D}$ in the image:

$$\begin{align*}
\text{maximize} & \quad \int \int f(x,y) dx dy \\
\text{subject to} & \quad \left| \hat{f}(\xi, \eta) \right| \leq \varepsilon \hat{f}(0,0), \\ & \quad f(x,y) = 0, \\ & \quad 0 \leq f(x,y) \leq 1, \quad \xi^2 + \eta^2 > 1,
\end{align*}$$

Here, $\varepsilon$ is a small positive constant (on the order of $10^{-5}$).

In general, the Fourier transform $\hat{f}$ is complex valued.

As formulated, this optimization problem has a linear objective function and both linear and second-order cone constraints.

Hence, a discretized version can be solved (to a global optimum).
Shaped Pupil Coronagraph (TPF-C)

20 petals

150 petals

Maybe We Can!
James Webb Space Telescope (JWST)

Throughput: ~ 45%
WA: 5 to 15 λ/D
Contrast: $10^{-5}$
Nancy Grace Roman Space Telescope

Repurposed NRO Spy Satellite

Similar to Hubble.
Aperture: 2.4 meters.
Central Obstruction and Spiders.