# Nonlinear Programming Models: A Survey

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# 1 Outline

- Convex Problems
  - Portfolio Optimization
  - Structural Optimization
  - Digital Audio Filters
  - Antenna-Array Design

- Nonconvex Problems
  - Putting on an Uneven Green
  - Range Maximization for a Hang Glider
  - Goddard Rocket Problem

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# 2 Convex vs. Nonconvex NLPs

## Nonlinear Programming (NLP)

$$
\begin{array}{lll}
\text{minimize} & f(x) & \\
\text{subject to} & h_i(x) = 0, & i \in \mathcal{E}, \\
& h_i(x) \geq 0, & i \in \mathcal{I}.
\end{array}
$$

NLP is convex if

- $h_i$'s in equality constraints are affine;
- $h_i$'s in inequality constraints are concave;
- $f$ is convex;

NLP is smooth if

- All are twice continuously differentiable.

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# 3 Portfolio Optimization

Start with historical data:

| Year | US 3-Month T-Bills | US Gov. Long Bonds | S&P 500 | Wilshire 5000 | NASDAQ Composite | Lehman Bros. Corp. Bonds | EAFE | Gold |
|---|---|---|---|---|---|---|---|---|
| 1973 | 1.075 | 0.942 | 0.852 | 0.815 | 0.698 | 1.023 | 0.851 | 1.677 |
| 1974 | 1.084 | 1.020 | 0.735 | 0.716 | 0.662 | 1.002 | 0.768 | 1.722 |
| 1975 | 1.061 | 1.056 | 1.371 | 1.385 | 1.318 | 1.123 | 1.354 | 0.760 |
| 1976 | 1.052 | 1.175 | 1.236 | 1.266 | 1.280 | 1.156 | 1.025 | 0.960 |
| 1977 | 1.055 | 1.002 | 0.926 | 0.974 | 1.093 | 1.030 | 1.181 | 1.200 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 1987 | 1.061 | 0.925 | 1.052 | 1.023 | 0.959 | 1.023 | 1.246 | 1.244 |
| 1988 | 1.071 | 1.086 | 1.165 | 1.179 | 1.165 | 1.076 | 1.283 | 0.861 |
| 1989 | 1.087 | 1.212 | 1.316 | 1.292 | 1.204 | 1.142 | 1.105 | 0.977 |
| 1990 | 1.080 | 1.054 | 0.968 | 0.938 | 0.830 | 1.083 | 0.766 | 0.922 |
| 1991 | 1.057 | 1.193 | 1.304 | 1.342 | 1.594 | 1.161 | 1.121 | 0.958 |

Notation: $R_j(t) =$ return on investment $j$ in time period $t$.

# 4 Portfolio Optimization—Continued

$$\text{maximize } \mathbf{E}u\left(\sum_j x_j \mathbf{R}_j\right) \approx \frac{1}{T}\sum_{t=1}^{T} u\left(\sum_j x_j R_j(t)\right)$$

$$\begin{aligned} \text{subject to } \sum_j x_j &= 1 \\ x_j &\geq 0 \qquad \text{for all investments } j \end{aligned}$$

where

$$u(x) = \begin{cases} \frac{1}{1-\gamma} x^{1-\gamma} & \gamma \neq 1 \\ \log(x) & \gamma = 1 \end{cases}$$

| | time (secs) |
|---|---|
| LOQO | 0.11 |
| LANCELOT | 0.12 |
| SNOPT | 0.12 |

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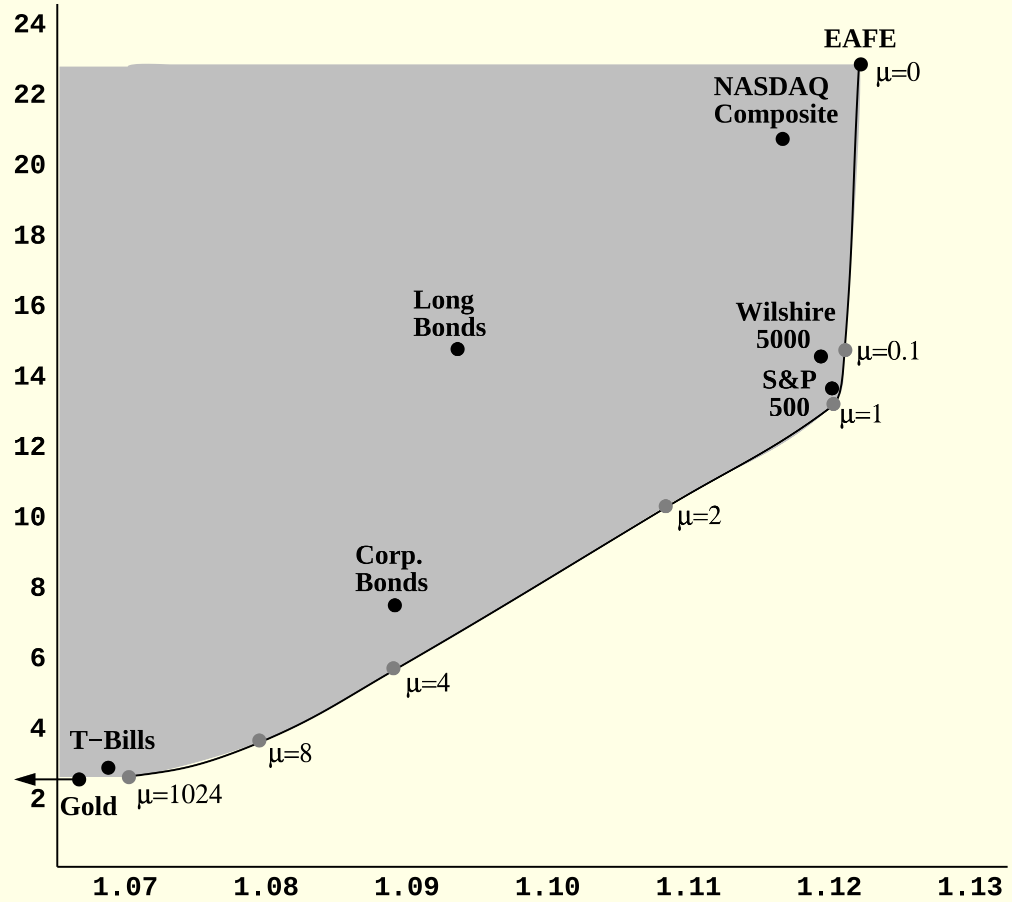
# 5 Portfolio Optimization—Continued

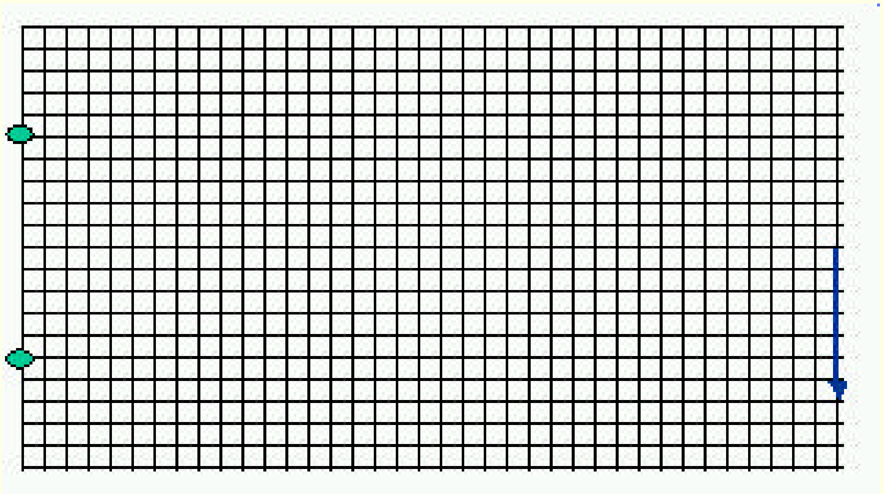
| $\gamma$ | Gold | US 3-Month T-Bills | Lehman Bros. Corp. Bonds | NASDAQ Composite | S&P 500 | EAFE | Mean | Std. Dev. |
|---|---|---|---|---|---|---|---|---|
| 0.088 | | | | | | 1.000 | 1.122 | 0.227 |
| 0.177 | | | | | 0.603 | 0.397 | 1.121 | 0.147 |
| 0.707 | | | | | 0.876 | 0.124 | 1.120 | 0.133 |
| 5.657 | | 0.036 | 0.322 | | 0.549 | 0.092 | 1.108 | 0.102 |
| 11.31 | | 0.487 | 0.189 | | 0.261 | 0.062 | 1.089 | 0.057 |
| 22.63 | | 0.713 | 0.123 | | 0.117 | 0.047 | 1.079 | 0.037 |
| 100.0 | 0.008 | 0.933 | 0.022 | 0.016 | | 0.022 | 1.070 | 0.028 |

# 6 Portfolio Optimization—Efficient Frontier

Plot of risk (i.e., std. dev.) vs. reward (i.e., mean return):



# 7 Structural Design



Given:

- A region of space in which to build something.
- Thing is essentially planar but with varying thickness.
- A place (or places) where the thing will be anchored.
- A place (or places) where loads will be applied.
- A certain total amount of material out of which to build the thing.

Objective: Design the thing to be as strong as possible.

Approach:

- Partition 2-D region into finite elements.
- Assign a thickness to each element.

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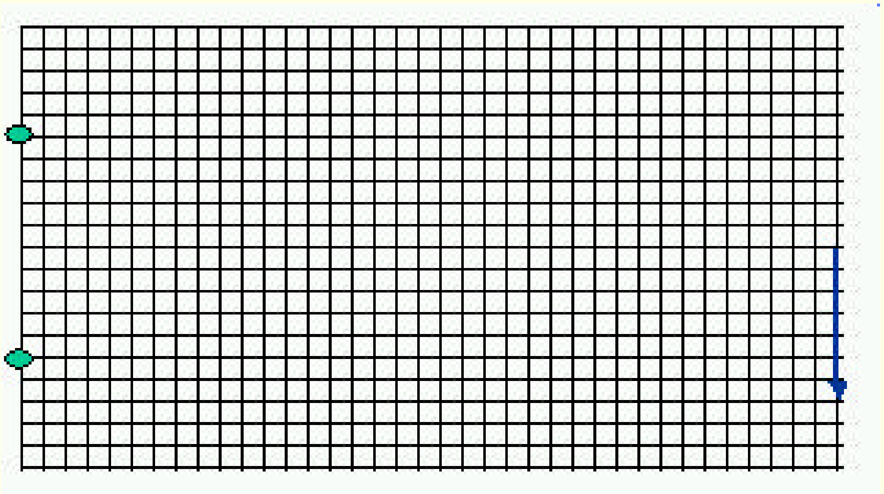

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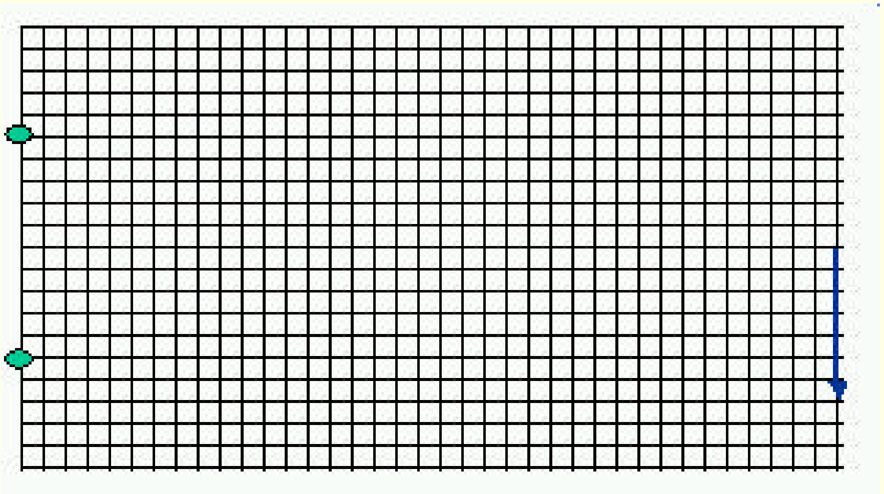

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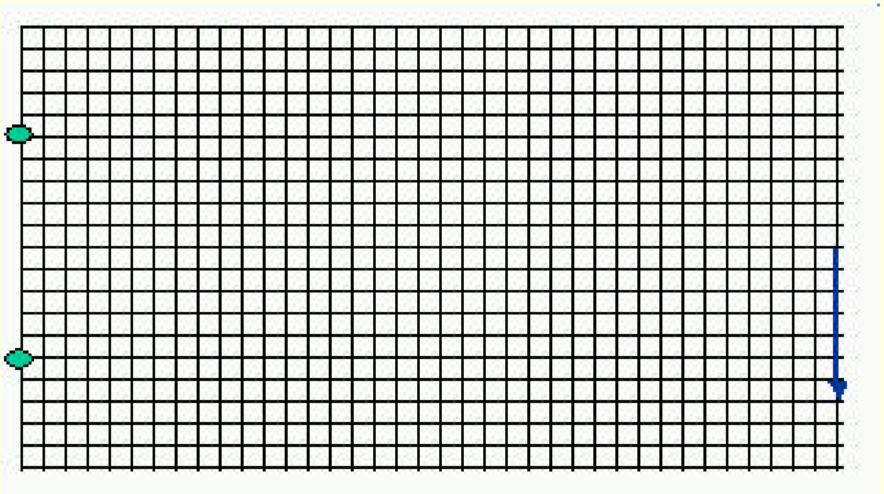

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# 8 Structural Design—Continued

$$\text{minimize} \quad -p^T w$$

$$\text{subject to} \quad \frac{V}{A_e} w^T K_e w \le 1, \quad e \in \mathcal{E}$$

where

$$\begin{aligned} p &= \text{applied load} \\ w &= \text{node displacements; } \textbf{optimization vars} \\ V &= \text{total volume} \\ A_e &= \text{thickness of element } e \\ K_e &= \text{element stiffness matrix } (\succeq 0) \\ \mathcal{E} &= \text{set of elements} \end{aligned}$$

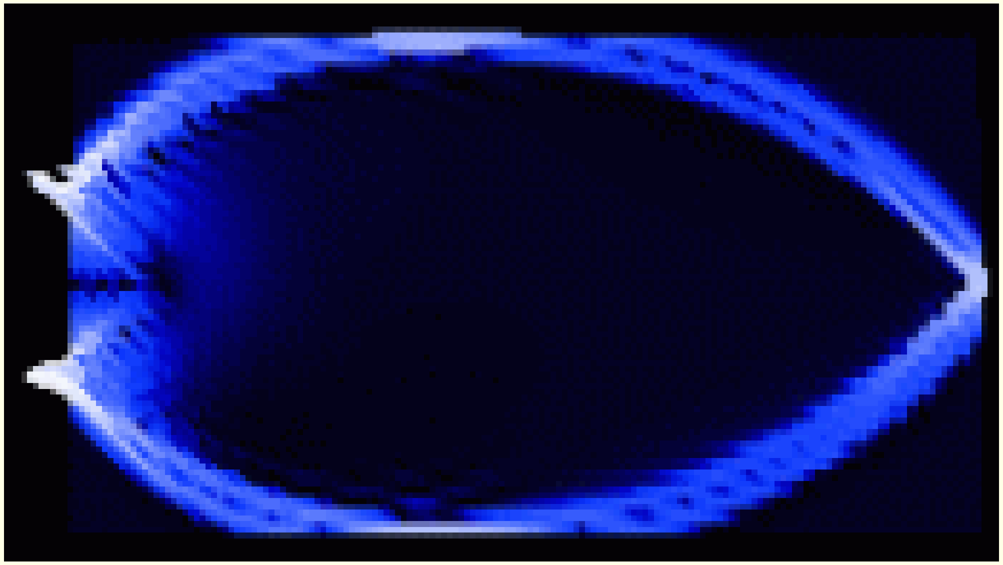
# 9 Specific Example: Michel Bracket

| element grid | 40x72 | 20x36 | 5x9 |
|---|---|---|---|
| constraints | 2880 | 720 | 45 |
| variables | 5965 | 1536 | 112 |

time (secs)

| | | | |
|---|---|---|---|
| LOQO | 412 | 89.7 | 2.32 |
| MINOS | $\infty$ | (IL) | (BS) |
| LANCELOT | $\infty$ | $\infty$ | 15.73 |
| SNOPT | - | (IS) | (BS) |

# 10 Solution

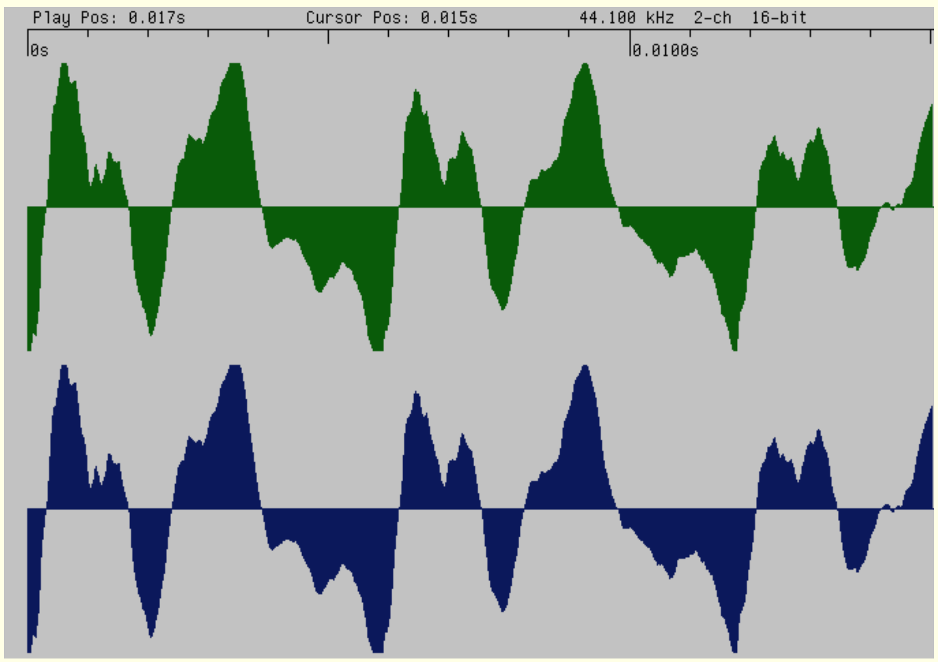


# 11 Finite Impulse Response (FIR) Filter Design

- Audio is stored digitally in a computer as a stream of short integers: $u_k$, $k \in \mathbb{Z}$.

- When the music is played, these integers are used to drive the displacement of the speaker from its resting position.

- For CD quality sound, 44100 short integers get played per second per channel.




| | | | | | |
|---|---|---|---|---|---|
| 0 | -32768 | 8 | -23681 | 16 | 12111 |
| 1 | -32768 | 9 | -18449 | 17 | 17311 |
| 2 | -32768 | 10 | -11025 | 18 | 21311 |
| 3 | -30753 | 11 | -6913 | 19 | 23055 |
| 4 | -28865 | 12 | -4337 | 20 | 23519 |
| 5 | -29105 | 13 | -1329 | 21 | 25247 |
| 6 | -29201 | 14 | 1743 | 22 | 27535 |
| 7 | -26513 | 15 | 6223 | 23 | 29471 |

# FIR Filter Design—Continued

- A finite impulse response (FIR) filter takes as input a digital signal and convolves this signal with a finite set of fixed numbers $h_{-n}, \ldots, h_n$ to produce a filtered output signal:

$$y_k = \sum_{i=-n}^{n} h_i u_{k-i}.$$

- Sparing the details, the output power at frequency $\nu$ is given by

$$|H(\nu)|$$

where

$$H(\nu) = \sum_{k=-n}^{n} h_k e^{2\pi i k \nu},$$

- Similarly, the mean squared deviation from a flat frequency response over a frequency range, say $\mathcal{L} \subset [0, 1]$, is given by

$$\frac{1}{|\mathcal{L}|} \int_{\mathcal{L}} \left| H(\nu) - 1 \right|^2 d\nu$$

# 13 Filter Design: Woofer, Midrange, Tweeter

minimize $\rho$

subject to
$$\int_0^1 \left(H_w(\nu) + H_m(\nu) + H_t(\nu) - 1\right)^2 d\nu \le \epsilon$$

$$\left(\frac{1}{|W|}\int_W H_w^2(\nu)d\nu\right)^{1/2} \le \rho \qquad W = [.2, .8]$$

$$\left(\frac{1}{|M|}\int_M H_m^2(\nu)d\nu\right)^{1/2} \le \rho \qquad M = [.4, .6] \cup [.9, .1]$$

$$\left(\frac{1}{|T|}\int_T H_t^2(\nu)d\nu\right)^{1/2} \le \rho \qquad T = [.7, .3]$$

where

$$H_i(\nu) = h_i(0) + 2\sum_{k=1}^{n-1} h_i(k)\cos(2\pi k\nu), \quad i = W, M, T$$

$h_i(k)$ = filter coefficients, i.e., **decision variables**

# Specific Example

$$\text{filter length: } n = 14$$
$$\text{integral discretization: } N = 1000$$

| | |
|---|---|
| constraints | 4 |
| variables | 43 |
| time (secs) | |
| LOQO | 79 |
| MINOS | 164 |
| LANCELOT | 3401 |
| SNOPT | 35 |

Ref: J.O. Coleman, U.S. Naval Research Laboratory,

CISS98 paper available: engr.umbc.edu/~jeffc/pubs/abstracts/ciss98.html

Click here for demo

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# 15 Antenna Array Design

- Given: a 2-D array of radar antennae.
- An incoming signal produces a signal at each antenna.
- A linear combination of the signals is made to produce one output signal.
- Coefficients of the linear combination can be chosen to accentuate and/or attenuate the output signal's strength as a function of the input signal's source direction.
- Similar to FIR filter design.
- The set of antennae is analogous to the set of time delays in FIR filter design.
- The direction of the input signal is analogous to frequency in FIR filter design.

# 2-D Antenna-Array Design Problem

$$\begin{array}{ll} \text{minimize} & \rho \\ \text{subject to} & |A(p)|^2 \le \rho, \quad p \in S \\ & A(p_0) = 1, \end{array}$$

where

$$\begin{aligned} A(p) &= \sum_{l \in \{\text{array elements}\}} w_l e^{-2\pi i p \cdot l}, \quad p \in S \\ w_l &= \text{complex-valued } \textbf{design weight} \text{ for array element } l \\ S &= \text{subset of unit hemisphere: sidelobe directions} \\ p_0 &= \text{“look” direction} \end{aligned}$$

# Specific Example: Hexagonal Lattice of 61 Elements

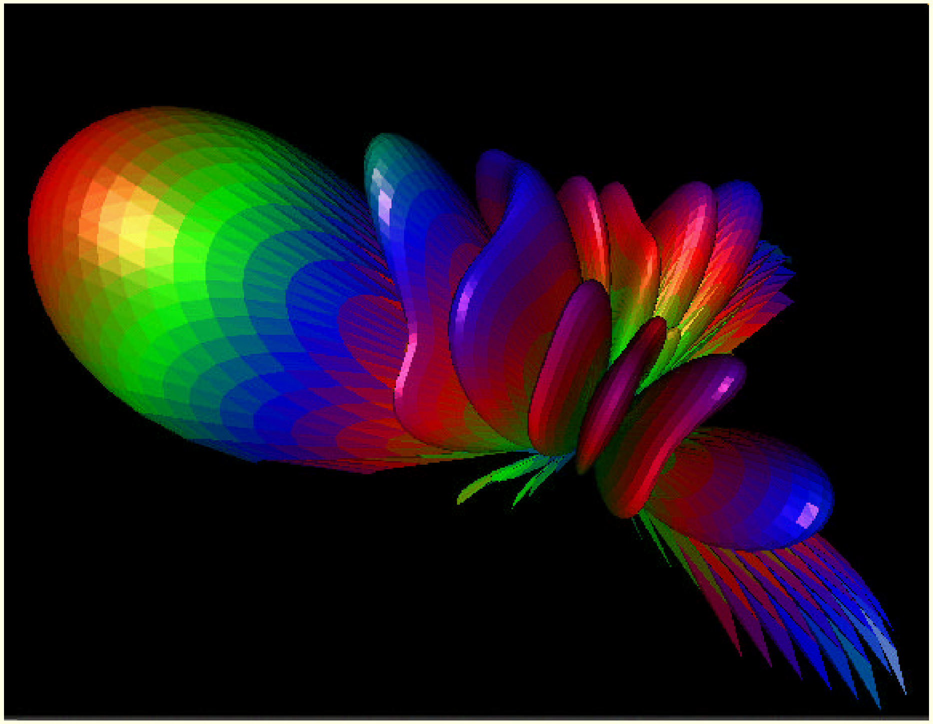
$$\rho = -20 \text{ dB} = 0.01$$

$$S = \text{889 points outside } 20^\circ \text{ from look direction}$$

$$p_0 = 40^\circ \text{ from zenith}$$

| | |
|---|---|
| constraints | 839 |
| variables | 123 |
| time (secs) | |
| LOQO | 722 |
| MINOS | $> 60000$ |
| LANCELOT | 55462 |
| SNOPT | — |

# 18 Solution



# 19 Putting on an Uneven Green

Given:

- $z(x, y)$ elevation of the green.
- Starting position of the ball $(x_0, y_0)$.
- Position of hole $(x_f, y_f)$.
- Coefficient of friction $\mu$.

Find: initial velocity vector so that ball will roll to the hole and arrive with minimal speed.

Variables:

- $u(t) = (x(t), y(t), z(t))$—position as a function of time $t$.
- $v(t) = (v_x(t), v_y(t), v_z(t))$—velocity.
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- $T$—time at which ball arrives at hole.

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# 20 Putting—Two Approaches

- Problem can be formulated with two decision variables:

$$v_x(0) \quad \text{and} \quad v_y(0)$$

  and two constraints:

$$x(T) = x_f \quad \text{and} \quad y(T) = y_f.$$

  In this case, $x(T)$, $y(T)$, and the objective function are complicated functions of the two variables that can only be computed by integrating the appropriate differential equation.

- A discretization of the complete trajectory (including position, velocity, and acceleration) can be taken as variables and the physical laws encoded in the differential equation can be written as constraints.

To implement the first approach, one would need an ode integrator that provides, in addition to the quantities being sought, first and possibly second derivatives of those quantities with respect to the decision variables.

The modern trend is to follow the second approach.

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# 21 Putting—Continued

Objective:

$$\text{minimize } v_x(T)^2 + v_y(T)^2.$$

Constraints:

$$\begin{aligned} v &= \dot{u} \\ a &= \dot{v} \\ ma &= N + F - mge_z \\ u(0) &= u_0 \qquad u(T) = u_f, \end{aligned}$$

where

- $m$ is the mass of the golf ball.
- $g$ is the acceleration due to gravity.
- $e_z$ is a unit vector in the positive $z$ direction.

and ...

# 21 Putting—Continued

Objective:

$$\text{minimize } v_x(T)^2 + v_y(T)^2.$$

Constraints:

$$\begin{aligned} v &= \dot{u} \\ a &= \dot{v} \\ ma &= N + F - mg e_z \\ u(0) &= u_0 \qquad u(T) = u_f, \end{aligned}$$

where

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where

- $m$ is the mass of the golf ball.
- $g$ is the acceleration due to gravity.
- $e_z$ is a unit vector in the positive $z$ direction.

and ...

# Putting—Continued

- $N = (N_x, N_y, N_z)$ is the normal force:

$$N_z = m\frac{g - a_x(t)\frac{\partial z}{\partial x} - a_y(t)\frac{\partial z}{\partial y} + a_z(t)}{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}$$

$$N_x = -\frac{\partial z}{\partial x}N_z$$

$$N_y = -\frac{\partial z}{\partial y}N_z.$$

- $F$ is the force due to friction:

$$F = -\mu\|N\|\frac{v}{\|v\|}.$$

# Putting—Continued

- $N = (N_x, N_y, N_z)$ is the normal force:

$$\begin{aligned} N_z &= m\frac{g - a_x(t)\frac{\partial z}{\partial x} - a_y(t)\frac{\partial z}{\partial y} + a_z(t)}{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1} \\ N_x &= -\frac{\partial z}{\partial x} N_z \\ N_y &= -\frac{\partial z}{\partial y} N_z. \end{aligned}$$

- $F$ is the force due to friction:

$$F = -\mu \|N\| \frac{v}{\|v\|}.$$

# Putting—Continued

- $N = (N_x, N_y, N_z)$ is the normal force:

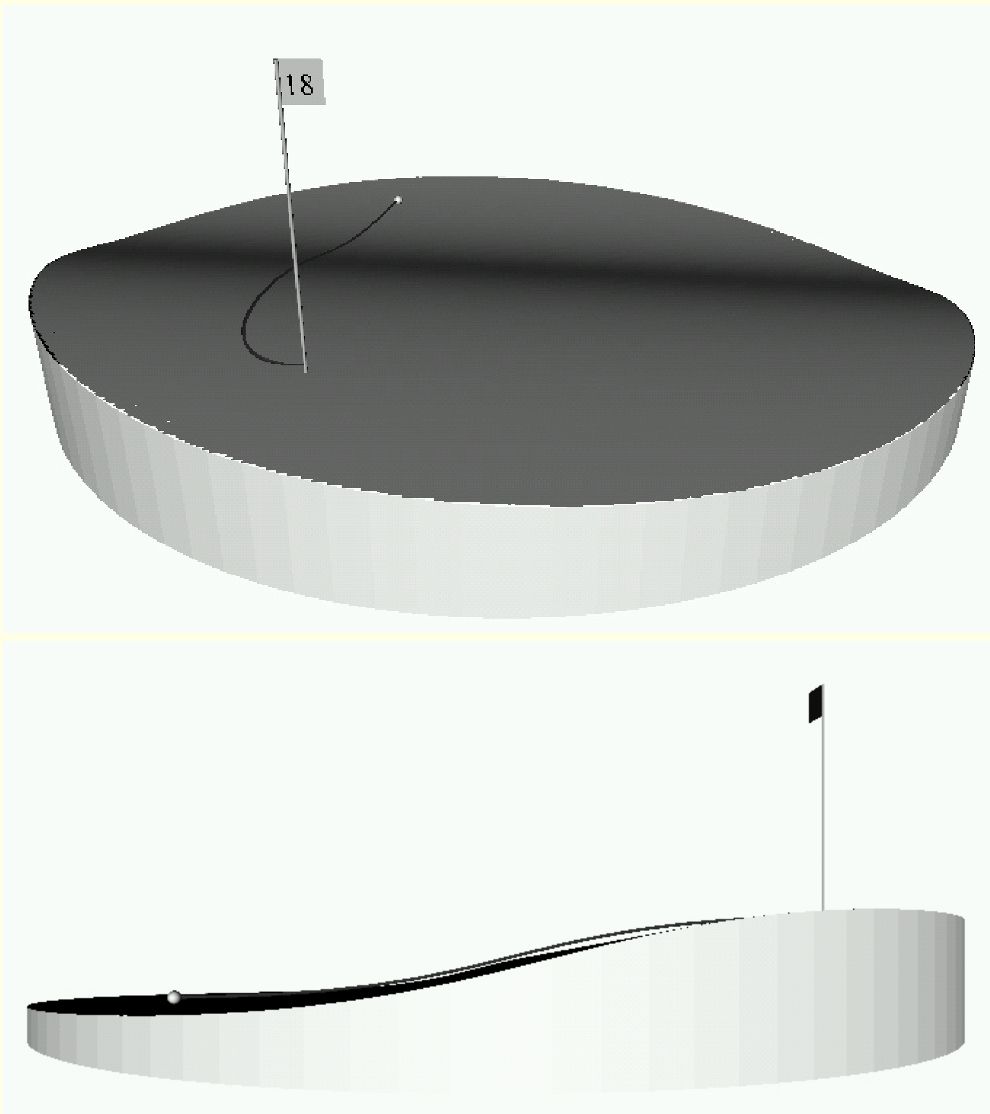
$$
\begin{aligned}
N_z &= m\frac{g - a_x(t)\frac{\partial z}{\partial x} - a_y(t)\frac{\partial z}{\partial y} + a_z(t)}{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1} \\
N_x &= -\frac{\partial z}{\partial x} N_z \\
N_y &= -\frac{\partial z}{\partial y} N_z.
\end{aligned}
$$

- $F$ is the force due to friction:

$$
F = -\mu \|N\| \frac{v}{\|v\|}.
$$

# Putting—Specific Example

- Discretize continuous time into $n = 200$ discrete time points.
- Use finite differences to approximate the derivatives.




| | |
|---|---|
| constraints | 597 |
| variables | 399 |
| time (secs) | |
| LOQO | 14.1 |
| LANCELOT | $> 600.0$ |
| SNOPT | 4.1 |

# Hang Gliding—Range Maximization

maximize $\quad x(T)$

subject to

$$v_x = \dot{x}, \qquad \dot{v}_x = \frac{1}{m}\left(-L\frac{v_y - u_a(x)}{v_r} - D\frac{v_x}{v_r}\right),$$

$$v_y = \dot{y}, \qquad \dot{v}_y = \frac{1}{m}\left(L\frac{v_x}{v_r} - D\frac{v_y - u_a(x)}{v_r}\right) - g$$

$$0 \le c_L \le c_{L\,\max}$$

where

$$\begin{aligned}
v_r &= \sqrt{v_x^2 + (v_y - u_a(x))^2}, \\
L &= \tfrac{1}{2} c_L \rho S v_r^2, \\
D &= \tfrac{1}{2} c_D(c_L) \rho S v_r^2 \\
c_D(c_L) &= c_0 + k c_L^2 \\
c_L &= \text{lift coefficient} \Longleftarrow \text{control variable} \\
u_a(x) &= \text{downrange updraft profile (see next slide)}
\end{aligned}$$

and $m$, $S$, $\rho$, $g$, $c_{L\,\max}$, $c_0$, and $k$ are constants. Everything else varies with time $t$.

# Hang Gliding—Range Maximization

maximize $\quad x(T)$

subject to

$$v_x = \dot{x}, \qquad \dot{v}_x = \frac{1}{m}\left(-L\frac{v_y - u_a(x)}{v_r} - D\frac{v_x}{v_r}\right),$$

$$v_y = \dot{y}, \qquad \dot{v}_y = \frac{1}{m}\left(L\frac{v_x}{v_r} - D\frac{v_y - u_a(x)}{v_r}\right) - g$$

$$0 \le c_L \le c_{L\,\max}$$

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$$\begin{aligned}
v_r &= \sqrt{v_x^2 + (v_y - u_a(x))^2}, \\
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and $m$, $S$, $\rho$, $g$, $c_{L\,\max}$, $c_0$, and $k$ are constants. Everything else varies with time $t$.

# Hang Gliding—Range Maximization

maximize $x(T)$

subject to

$$v_x = \dot{x}, \qquad \dot{v}_x = \frac{1}{m}\left(-L\frac{v_y - u_a(x)}{v_r} - D\frac{v_x}{v_r}\right),$$

$$v_y = \dot{y}, \qquad \dot{v}_y = \frac{1}{m}\left(L\frac{v_x}{v_r} - D\frac{v_y - u_a(x)}{v_r}\right) - g$$

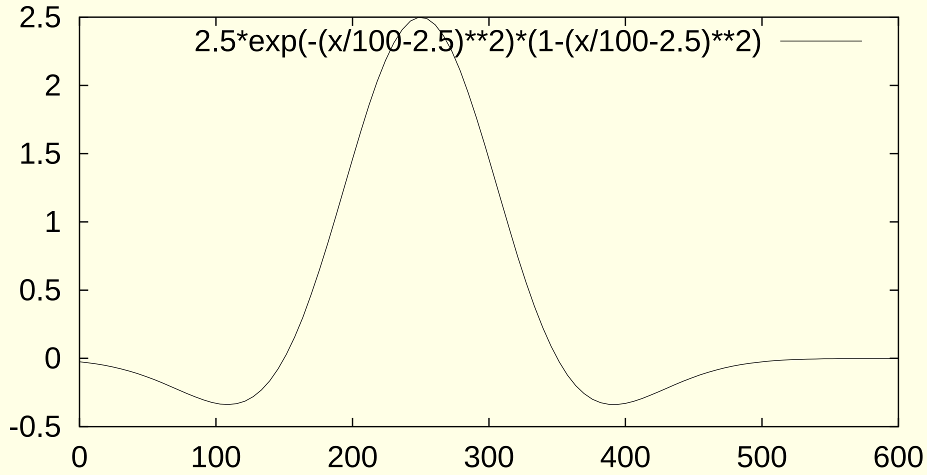
$$0 \le c_L \le c_{L\,\max}$$

where

$$\begin{aligned} v_r &= \sqrt{v_x^2 + (v_y - u_a(x))^2}, \\ L &= \tfrac{1}{2} c_L \rho S v_r^2, \\ D &= \tfrac{1}{2} c_D(c_L) \rho S v_r^2 \\ c_D(c_L) &= c_0 + k c_L^2 \\ c_L &= \text{lift coefficient} \Longleftarrow \text{control variable} \\ u_a(x) &= \text{downrange updraft profile (see next slide)} \end{aligned}$$

and $m$, $S$, $\rho$, $g$, $c_{L\,\max}$, $c_0$, and $k$ are constants. Everything else varies with time $t$.

# Hang Gliding—Updraft Profile



# Hang Gliding—Optimal Trajectory

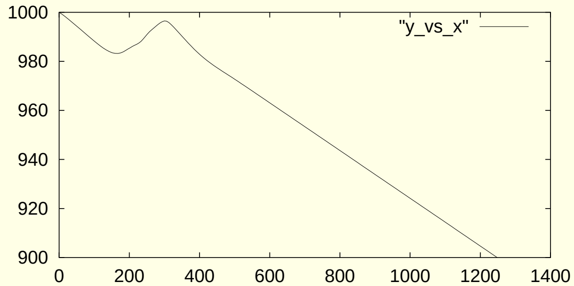



Figure 1: $y$ vs $x$

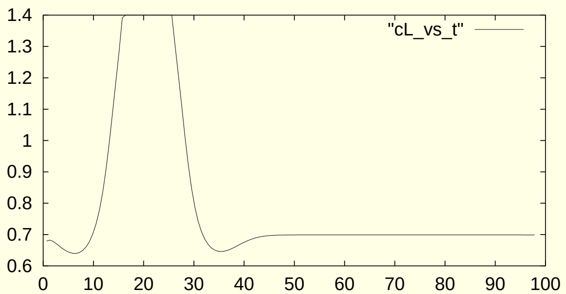



Figure 2: $cL$ vs $t$

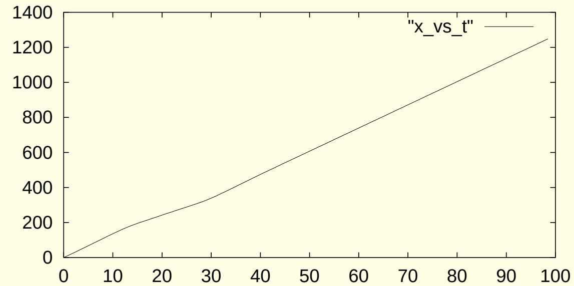



Figure 3: $x$ vs $t$

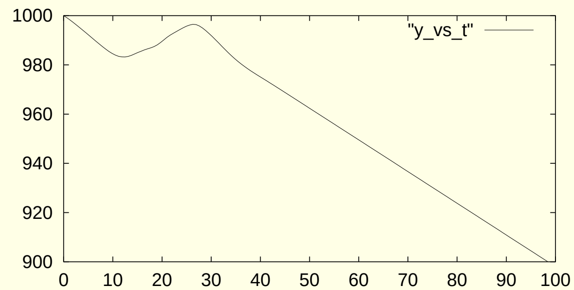



Figure 4: $y$ vs $t$

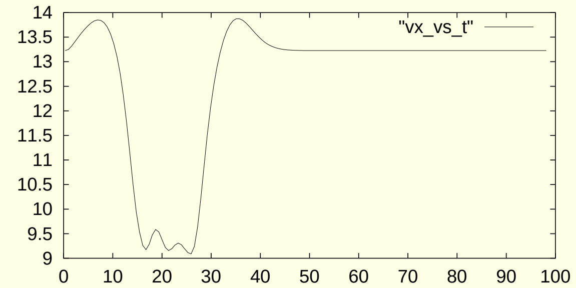



Figure 5: $v_x$ vs $t$

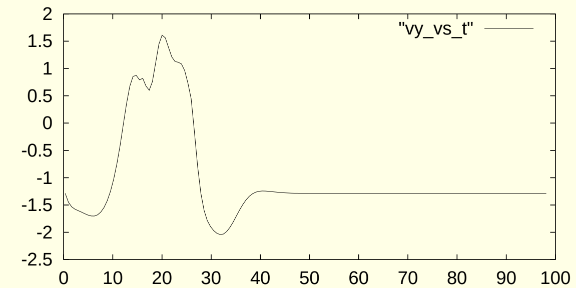



Figure 6: $v_y$ vs $t$

# Hang Gliding—Solver Comparison

| | |
|---|---|
| constraints | 600 |
| variables | 449 |
| time (secs) | |
| LOQO | 10.7 |
| LANCELOT | $> 3600$ |
| SNOPT | 244.1 |

# 28 Goddard Rocket Problem

Objective:

$$\text{maximize } h(T);$$

Constraints:

$$
\begin{aligned}
v &= \dot{h} \\
a &= \dot{v} \\
\theta &= -c\dot{m} \\
ma &= (\theta - \sigma v^2 e^{-h/h_0}) - gm \\
0 &\leq \theta \leq \theta_{\max} \\
m &\geq m_{\min} \\
h(0) &= 0 \qquad v(0) = 0 \qquad m(0) = 3
\end{aligned}
$$

where

- $\theta =$ Thrust , $m =$ mass
- $\theta_{\max}$, $g$, $\sigma$, $c$, and $h_0$ are given constants
- $h$, $v$, $a$, $T_h$, and $m$ are functions of time $0 \leq t \leq T$.

# 28 Goddard Rocket Problem

Objective:

$$\text{maximize } h(T);$$

Constraints:

$$\begin{aligned}
v &= \dot{h} \\
a &= \dot{v} \\
\theta &= -c\dot{m} \\
ma &= (\theta - \sigma v^2 e^{-h/h_0}) - gm \\
0 &\leq \theta \leq \theta_{\max} \\
m &\geq m_{\min} \\
h(0) &= 0 \qquad v(0) = 0 \qquad m(0) = 3
\end{aligned}$$

where

- $\theta =$ Thrust , $m =$ mass
- $\theta_{\max}$, $g$, $\sigma$, $c$, and $h_0$ are given constants
- $h$, $v$, $a$, $T_h$, and $m$ are functions of time $0 \leq t \leq T$.

# Goddard Rocket Problem—Solution

| | |
|---|---|
| constraints | $\mathbf{399}$ |
| variables | $\mathbf{599}$ |
| time (secs) | |
| LOQO | $\mathbf{5.2}$ |
| LANCELOT | $\boldsymbol{(IL)}$ |
| SNOPT | $\boldsymbol{(IL)}$ |