Uniform Continuity is Almost Lipschitz Continuity *

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Abstract

We prove that uniformly continuous functions on convex sets are almost Lipschitz continuous in the sense that f is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a $K < \infty$, such that $\|f(y) - f(x)\| \leq K\|y - x\| + \epsilon$.

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In this paper, we are interested in the relationship between two important subclasses of continuous functions: uniformly-continuous

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functions and Lipschitz-continuous functions. Throughout, we assume that all functions map a domain D of one normed linear space into another.

Recall the definition of uniform continuity:

Definition 1 *A function f is uniformly continuous if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that $\|f(y) - f(x)\| < \epsilon$ whenever $\|y - x\| < \delta$.*

The definition of Lipschitz continuity is also familiar:

Definition 2 *A function f is Lipschitz continuous if there exists a $K < \infty$ such that $\|f(y) - f(x)\| \leq K\|y - x\|$.*

It is easy to see (and well-known) that Lipschitz continuity is a stronger notion of continuity than uniform continuity. For example, the function $f(x) = x^{1/3}$ on $\Re$ is uniformly continuous but not Lipschitz continuous. Hence, it is perhaps surprising to note that uniformly continuous functions are almost Lipschitz:

Theorem 1 *A function f defined on a convex domain is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a $K < \infty$ such that $\|f(y) - f(x)\| \leq K\|y - x\| + \epsilon$.*

Proof. Suppose that f is uniformly continuous on a convex domain D and fix $\epsilon > 0$. Then, there exists a $\delta > 0$ such that $\|f(z) - f(z')\| < \epsilon$ whenever $\|z - z'\| < \delta$. Fix x and y in D and let

$$z_k = x + k\frac{\delta}{2}\frac{y - x}{\|y - x\|} \quad \text{for } k = 0, 1, 2, \ldots, N$$

where

$$N = \lfloor \frac{\|y - x\|}{\delta/2} \rfloor,$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function. From the convexity of D, we see that each z_k belongs to D. Also,

$$z_0 = x,$$

$$\|z_k - z_{k-1}\| = \delta/2,$$

and

$$\|y - z_N\| < \delta/2.$$

Hence,

$$\begin{aligned}\|f(y) - f(x)\| &\leq \sum_{k=1}^{N} \|f(z_k) - f(z_{k-1})\| + \|f(y) - f(z_N)\| \\ &< (N+1)\epsilon \\ &\leq \frac{2\epsilon}{\delta}\|y - x\| + \epsilon.\end{aligned}$$

Picking $K = 2\epsilon/\delta$ establishes the "only if" direction.

For the "if" part, suppose that f satisfies the condition given in the theorem. Fix $\epsilon > 0$ and choose K so that

$$\|f(y) - f(x)\| \leq K\|y - x\| + \epsilon/2$$

for all $x, y \in D$. Put $\delta = \epsilon/2K$. If $\|y - x\| < \delta$, then we see that

$$\|f(y) - f(x)\| < \epsilon.$$

Hence, f is uniformly continuous. □

The author originally discovered this result when puzzling over why a uniformly continuous function can have unbounded slopes (such as in the cube root example), but cannot grow faster than linearly. The ϵ in our "almost Lipschitz" characterization allows for unbounded slopes locally, but nevertheless, it is now easy to see that the growth at infinity is at most linear. Indeed, suppose that 0 is a point in D, and apply our characterization to 0 and y to see that

$$\begin{aligned}\|f(y)\| &\leq \|f(0)\| + \|f(y) - f(0)\| \\ &\leq \|f(0)\| + K\|y\| + \epsilon\end{aligned}$$

which shows that $\|f(y)\| \le a + b\|y\|$ for an appropriate choice of a and b.

To illustrate the use of this new characterization of uniform continuity, we reprove one of the standard theorems (see e.g. [Roy88]) about uniformly continuous functions:

Theorem 2 *If* $\{x_n : n = 0, 1, 2, \ldots\}$ *is a Cauchy sequence and* f *is uniformly continuous (on a convex domain* D*), then* $\{f(x_n) : n = 0, 1, 2, \ldots\}$ *is also a Cauchy sequence.*

Proof. Fix an arbitrary $\epsilon > 0$. Since f is uniformly continuous, there exists a constant $K < \infty$ such that

$$\|f(x_n) - f(x_m)\| \le K\|x_n - x_m\| + \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence, $\lim_{N\to\infty} \sup_{n,m\ge N} \|x_n - x_m\| = 0$. Hence,

$$0 \le \lim_{N\to\infty} \sup_{n,m\ge N} \|f(x_n) - f(x_m)\| \le \epsilon.$$

But ϵ was arbitrary and so the lim sup actually vanishes which proves that $\{f(x_n)\}$ is a Cauchy sequence. □

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References

[Roy88] H.L. Royden. *Real Analysis.* Macmillan Publishing Company, 1988.